Ph.D. Examination – Topology
August 2020

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Let X be a topological space. Given two points $a, b \in X$, a collection of sets $A_1, \ldots, A_n$ is a *simple chain from a to b* if $a \in A_1$, $b \in A_n$, and $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Prove that if $\{U_\alpha\}$ is a collection of open sets covering X and X is connected, then there is a simple chain of elements in $\{U_\alpha\}$ joining any pair of points $a, b \in X$. (Hint: Consider the set C_a of all b so that there is a simple chain of elements joining a and b. Show that C_a is both open and closed.)

2. Let $X = S^2 \cup_f e^3 \cup_h e^5$, where $f : S^2 \to S^2$ has degree n and $h : S^4 \to S^2 \cup_f e^3$ is continuous.
   (a) What is the Euler characteristic of X?
   (b) Determine $H_\bullet(X; \mathbb{Z})$ and $H^\bullet(X; \mathbb{Z}_2)$ as a function of n.
   (c) Can X have the homotopy type of a closed 5-manifold? (If the answer depends on n, explain.)

3. Let X be the space $\mathbb{R}P^3/\mathbb{R}P^1$ (the space obtained by collapsing $\mathbb{R}P^1 \subset \mathbb{R}P^3$ to a point). Compute the integral homology groups of X.

4. Let $\mathbb{R}P^2$ be the real projective plane and fix a basepoint $x_0 \in \mathbb{R}P^2$. Note that we can realize the wedge sum $\mathbb{R}P^2 \vee \mathbb{R}P^2$ as $(\mathbb{R}P^2 \times \{x_0\}) \cup (\{x_0\} \times \mathbb{R}P^2) \subset \mathbb{R}P^2 \times \mathbb{R}P^2$.
   (a) Compute the fundamental groups of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.
   (b) Show that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
   (c) Show that any map $f : \mathbb{R}P^2 \vee \mathbb{R}P^2 \to S^1$ is null-homotopic.

5. Compute $H_\bullet(M; \mathbb{Z}_2)$, where M is the space with integral homology groups

$$H_k(M; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 2, 4 \\ \mathbb{Z}_2 & k = 1 \\ \mathbb{Z}_3 & k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Answer the following with complete definitions, statements, or short proofs.

6. Let M be an orientable simply-connected 3-manifold. Compute the integral homology of M.

7. Compute $\chi(\mathbb{R}P^2 \times \mathbb{C}P^4 \times S^4)$

8. Give an example of a space that is connected but not locally connected.

9. State the Urysohn Lemma.

10. Prove that if $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

11. Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{N}^*$ be its one-point compactification. Let X be any topological space. Prove that a sequence $\{a_n\}$ in X converges to a if and only if the map $f : \mathbb{N}^* \to X$ defined by $f(n) = a_n$ and $f(\infty) = a$ is continuous. (Here $\mathbb{N}$ has the discrete topology.)

12. State the Borsuk-Ulam Theorem.

13. Describe the possible connected covering spaces of $S^1 \times S^1$.

14. Does the following exact sequence of abelian groups necessarily split? Prove or give a counterexample.

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to A \to 0$$

15. Compute the integral homology of the space $\mathbb{C}P^2 \times \mathbb{R}P^3$.