Variational Analysis/Numerical Optimization PhD Qualifier – January 2026

Do 8 problems, 4 from problems 1–5 and 4 from problems 6–10.

1. Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

   $$f[(1-\alpha)\mathbf{x} + \alpha\mathbf{y}] \le (1-\alpha)f(\mathbf{x}) + \alpha f(\mathbf{y})$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in [0, 1]$.

   (a) If f is continuously differentiable, then show that f is convex if and only if

   $$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

   (b) If f is twice continuously differentiable, then show that f is convex if and only if $\nabla^2 f$ is positive semidefinite on $\mathbb{R}^n$.

2. Given $\mathbf{x}_0 \in \mathbb{R}^n$, let $0 < \sigma < \mu < 1$ be Wolfe line search parameters. In a Wolfe line search for minimizing $f : \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x}_{k+1} = \mathbf{x}_k - s_k\mathbf{g}_k$ where $\mathbf{g}_k = \nabla f(\mathbf{x}_k)^T$ and the stepsize s_k at step k is chosen so that the following two conditions hold:

   W1. $F(s_k) \le F_L(s_k)$, where $F(s) = f(\mathbf{x}_k - s\mathbf{g}_k)$ and $F_L(s) = f(\mathbf{x}_k) - \sigma s\|\mathbf{g}_k\|^2$.
   W2. $F'(s_k) \ge \mu F'(0) = -\mu\|\mathbf{g}_k\|^2$.

   If f is bounded from below, prove that there exists a stepsize $s_k > 0$ satisfying the Wolfe conditions. If f is twice continuously differentiable over the convex hull of the level set

   $$L = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \le f(\mathbf{x}_0)\},$$

   and the largest eigenvalue of $\nabla^2 f$ is bounded by $M < \infty$ over L, then prove that the gradient g_k converges to zero.

3. Suppose $f : \mathcal{K} \to \mathbb{R}$ where $\mathcal{K} \subset \mathbb{R}^n$ is a convex set.

   (a) If $\mathbf{x}^*$ is a local minimizer of f, then show that

   $$\nabla f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \ge 0 \quad \text{for all } \mathbf{x} \in \mathcal{K}. \tag{1}$$

   (b) If f is convex over $\mathcal{K}$ and differentiable at some point $\mathbf{x}^* \in \mathcal{K}$ where (1) holds, then show that $\mathbf{x}^*$ is a global minimizer for f over $\mathcal{K}$.

   (c) Consider the quadratic programming problem

   $$\min \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} \quad \text{subject to } \mathbf{x} \in \mathcal{K},$$

   where $\mathcal{K} \subset \mathbb{R}^n$ is a closed convex set and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive definite with smallest eigenvalue $\alpha > 0$. Show that the solutions $\mathbf{u}_i$, $i = 1, 2$, associated with $\mathbf{b} = \mathbf{b}_i$, $i = 1, 2$ respectively, satisfy

   $$\|\mathbf{u}_1 - \mathbf{u}_2\| \le \|\mathbf{b}_1 - \mathbf{b}_2\|/\alpha.$$

4. Prove the arithmetic-geometric mean inequality:

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$

where $\mathbf{x} \geq \mathbf{0}$. *Hint*: change variables to $\mathbf{y} = c\mathbf{x}$ where c is a scalar chosen so that the components of y sum to 1. Note that maximizing the product of the y_i is equivalent to maximizing the log of the product since $\log(t)$ is a monotone increasing function of t on $(0, \infty)$. Using duality, the max of the log of the product can be determined explicitly.

5. Consider the quadratic program

$$\min \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} \quad \text{subject to } \mathbf{a}^T\mathbf{x} = \mathbf{c}, \tag{2}$$

where $\mathbf{A}$ is a symmetric, positive definite matrix. The penalty approximation to (2) is

$$\min \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{b}^T\mathbf{x} + \frac{p}{2}(\mathbf{a}^T\mathbf{x} - \mathbf{c})^2, \tag{3}$$

where $p > 0$ is the penalty parameter. Use the first-order optimality conditions for both (2) and (3) to obtain a bound for the distance between the solution of (2) and the solution of (3).

6. Let us consider the minimization problem

$$\min \ I(y) := \int_0^1 f(y'(x), y(x), x)\, dx \quad \text{subject to } y \in C_0^1, \tag{4}$$

where f is a given continuously differentiable function. If y^* is a local minimizer of this problem, s is a scalar, $h \in C_0^1$, then setting to zero the derivative of $I(y^* + sh)$ evaluated at $s = 0$ yields the following first-order optimality condition:

$$\int_0^1 \left[\left(\frac{\partial f}{\partial y'}\right)^*(x)h'(x) + \left(\frac{\partial f}{\partial y}\right)^*(x)h(x)\right] dx = 0 \tag{5}$$

for all $h \in C_0^1$, where

$$\left(\frac{\partial f}{\partial y'}\right)^*(x) := \frac{\partial f}{\partial y'}(y^{*\prime}(x), y^*(x), x) \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)^*(x) := \frac{\partial f}{\partial y}(y^{*\prime}(x), y^*(x), x).$$

(a) As a consequence of (5), show that

$$\left(\frac{\partial f}{\partial y'}\right)^* \in C^1 \tag{6}$$

and the Euler equation is satisfied. Note that it is only assumed that f is continuously differentiable; you should show that at a local minimizer of (4), the partial derivative in (6) is continuously differentiable.

(b) For the special case

$$f(y'(x), y(x), x) = \frac{1}{2}y'(x)^2 - \varphi(x)y(x)^2,$$

where $\varphi \in C^k$ for some integer $k \geq 0$, what does the observation in (a) imply concerning the smoothness of a minimizer y^* for (4)?

7. Suppose that P and Q are real-valued functions defined on the interval $[0, 1]$ with P continuously differentiable and Q continuous. It is assumed that there exists $\alpha > 0$ such that $P(x) \geq \alpha$ for all $x \in [0, 1]$. Define by

$$(h) := \int_0^1 P(x)h'(x)^2 + Q(x)h(x)^2 \, dx.$$

Suppose that has the following property: There exists > 0 such that

$$(h) \geq \int_0^1 h'(x)^2 + h(x)^2 \, dx \quad \text{for all } h \in H_0^1,$$

where H_0^1 is the space of functions with h' square integrable over $[0, 1]$ and $h(0) = h(1) = 0$.

Prove the following: If u is a solution of the boundary-value problem

$$\frac{d}{dx}(P(x)u'(x)) = Q(x)u(x) \quad \text{for all } x \in [0, a] \quad \text{with} \quad u(0) = 0 = u(a),$$

where $a \in (0, 1)$ is given, then u vanishes on the entire interval $[0, a]$. *Hint*: extend u from $[0, a]$ to $[0, 1]$ by setting $u(x) = 0$ for $x \in [a, 1]$. Then integrate by parts to show that $(u) = 0$.

8. Consider the variational problem

$$\min \int_0^1 \frac{1}{2}u'(x)^2 + e^{u(x)} \, dx \quad \text{subject to } u \in H_0^1. \tag{7}$$

(a) What is the first-order necessary optimality condition (Euler equation) for a local minimizer of this variational problem?

(b) Show that if a minimizer of (7) exists, then it is unique.

9. Consider the initial-value problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = 0,$$

where $x : [0, 1] \to \mathbb{R}^n$ and $\mathbf{A} \in L^\infty([0, 1]; \mathbb{R}^{n\times n})$. Obtain a bound for the sup-norm of $\mathbf{x}$ in terms of the L^2 norm of $\mathbf{u}$. *Hint*: Recall Grönwall's inequality

$$x(t) \leq \alpha(t) + L\int_0^t x(s)\, ds \text{ on } [0, a] \;\Rightarrow\; x(t) \leq \alpha(t)e^{Lt} \text{ on } [0, a]$$

where α is monotone nondecreasing.

10. Consider the following control problem:

$$\min \int_0^1 \frac{1}{2}u^2(x) + y(x)\,dx \quad \text{subject to } y'(x) = u(x),\ y(0) = 0,\ u(x) \geq \ell(x),$$

where $\ell(x)$ is a given lower bound for the control $u(x)$ at each $x \in [0,1]$.

(a) What is the first-order optimality condition (Pontryagin minimum principle) for this problem?

(b) What is the solution of the control problem?