First-Semester Algebra Exam **January 14, 2026**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let $n \geq 2$ and let G be a cyclic group of order n. Prove that $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. (Here $(\mathbb{Z}/n\mathbb{Z})^\times$ is the unit group of the ring $\mathbb{Z}/n\mathbb{Z}$.)

2. Prove that there is no simple group of order $9477 = 3^6 \cdot 13$. (Consider the action of G on the set of left cosets of a Sylow 3-subgroup of G.)

3. Construct four groups of order 28, and prove that they are pairwise non-isomorphic.

4. Let G be a group.

   (a) (3 points) Define the upper central series of G:
   $$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

   (b) (4 points) Prove that the groups $Z_i(G)$ are all characteristic subgroups of G.

   (c) (3 points) Give an example of a nontrivial group G such that the groups $Z_i(G)$ are all trivial.

5. Prove that the ring $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm $\mathrm{N}(a + b\sqrt{3}) = |a^2 - 3b^2|$.

6. For each of the following polynomials $f(X) \in \mathbb{Q}[X]$, either prove that $f(X)$ is irreducible or give a nontrivial factorization of $f(X)$.

   (a) (3 points) $f(X) = X^4 - 6X^2 + 30X - 90$

   (b) (4 points) $f(X) = X^4 + 5X^3 - 6X^2 - 23$

   (c) (3 points) $f(X) = X^4 + X^3 + X^2 + X + 1$