Study Problems for Topology Ph.D. Exam

1. Construct a map from I onto I^2.

2. Characterize those Hausdorff spaces X for which there is a map $f: I \to X$ which is onto.

3. Characterize those Hausdorff spaces X for which there is a map $f: C \to X$ which is onto where C is the Cantor set.

4. Prove that the fundamental group of S^1 is Z.

5. Show that the fundamental group of S^n is trivial for $n > 1$.

6. Show that $H_n(S^n; Z) = Z$ and that $H_0(S^n; Z) = Z$ for $n \geq 1$.

7. Let G be a finitely presented group. Construct a finite connected polyhedron P such that $\pi_1(P) = G$.

8. Show that $\pi_1(T^n) = Z^n$.

9. Show that $\pi_1(P_n) = Z_2$ for $n > 1$, where P_n is n-dimensional real projective space.

10. Show that $\pi_1\left(\bigvee_{i=1}^{n} S^1\right) = F_n$ where F_n is the free group on n generators.

11. Show that $\check{H}^1(\Sigma_2) = Q_2 = \left\{\frac{m}{2^n} \mid m \in Z, n \in N\right\}$ where Σ_2 is the dyadic Solenoid.

12. Show that there is no map from I onto the $\sin(1/x)$ – curve.

13. Let $f: I \to I$ be a map and let $\{J_i\}_{i=1}^{n}$ be a collection of subintervals in I such that each pair has at most endpoints in common. Let A be the set of endpoints and suppose $f(A) \subset A$. Define the Markov Graph for f with respect to $\{J_i\}_{i=1}^{n}$.

14. Let $f: I \to I$ be a map and let $\{J_i\}_{i=1}^{n}$ be a collection of subintervals in I such that each pair has at most endpoints in common. Let A be the set of endpoints and suppose $f(A) \subset A$. Show that for each cycle of length n in the Markov Graph, there is a point x in I such that $f^n(x) = x$ with the itinerary of x the given cycle.

15. Show that if $f: I \to I$ has a point with a period three orbit, then for every n, f has a point of period n.

16. Let X be connected, arcwise connected, and semilocally simply connected. Show that $\pi_1(X) \cong G$ where G is the group of deck transformations of the universal covering of X.

17. State and prove the Contraction Mapping Theorem.

18. State and prove the Brouwer Fixed Point Theorem.

19. State and prove the Baire Category Theorem.

20. Prove that a subset A of the real line R is connected if and only if it is an interval.

21. State and prove the Seifert-van Kampen Theorem.

22. State and prove the Hahn-Mazurkiewicz Theorem.

23. Show that if X is a locally connected connected compact metric space, then X is arcwise connected.

24. Show that if U is an open subset of R^n, then U is connected if and only if between any pair of points $x, y \in U$ there is a polygonal arc in U going from x to y.

25. Let $f: S^n \to S^n$ be the antipodal map. Show that for n even, f is not homotopic to the identity map on S^n. Show that for n odd, f is homotopic to the identity map.

26. State the Fundamental Theorem of Algebra and give a topological proof.

27. Let G be a topological group whose underlying space is a connected n-manifold. Show that the covering space of G, $\tilde{G}$, also has this property.

28. Show that if a metric space X is pathwise connected, then it is arcwise connected.

29. Determine the arc components of the dyadic solenoid.

30. Let X be any compact metric space. Show that there is a continuous map $f: C \to X$ which is onto where C is the Cantor set.

31. Let X be an indecomposable metric continuum. Show that there are uncountably many composants and that these are pairwise disjoint.

32. Let X be a metric continuum. Show that X is indecomposable if and only if X does not contain any proper subcontinuum with nonempty interior.

33. Show that the dyadic solenoid is indecomposable.

34. Show that a metric space is normal.

35. Give the Mayer-Vietoris sequence for homology. Use this to compute the homology groups for the n-sphere S^n.

36. Classify the compact surfaces.

37. [Urysohn Lemma] Show that if X is a normal space and A and B are disjoint closed subsets of X, then there is a continuous function $f: X \to [0,1]$ having the property that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

38. [Tietze Extension Theorem] Let X be a normal space and A a closed subset of X. Suppose that $f: A \to [0,1]$ is continuous. Show that there is a continuous $F: X \to [0,1]$ so that the following diagram commutes.

$$\begin{array}{ccc} A & \subset & X \\ \downarrow f & \nearrow F & \\ [0,1] & & \end{array}$$

39. [Five Lemma]. Consider the following commutative diagram of abelian groups and homomorphisms. Suppose that the rows are exact in the diagram. Show that if $\{f_1, f_2, f_4, f_5\}$ are isomorphisms, then so is f_3.

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

40. Show that there is no continuous map $r: B^{n+1} \to \partial B^{n+1} = S^n$ with $r(x) = x$ for every $x \in \partial B^{n+1}$.

41. Define the mapping cylinder of $f: X \to Y$.

42. Define the mapping torus of $f: X \to X$.

43. Define the cone of X. Define the suspension of X.

44. Show that there is a nontrivial flow, $F: R \times T_f \to T_f$, on the mapping torus of $f: X \to X$ such that the time one map of F on X corresponds to f.

45. Let ΣX be the two-point suspension of X. Show that $H_{k+1}(\Sigma X) \cong H_k(X)$.

46. Suppose X is an arc connected, locally arc connected, and locally simply connected. Suppose that $\pi_1(X) = G$ and that $\varphi: G \to H$ is an onto group homomorphism. Show that there is a covering space of X whose collection of covering transformations is isomorphic to H and whose fundamental group is isomorphic to the kernel of φ.

47. Let $0 \to A \to B \to C \to 0$ be an exact sequence of abelian groups. If C is a free group, show that $B \cong A \oplus C$.

48. Compute the fundamental group of the Dunces hat.

49. Compute the fundamental group of the mapping torus of $f: S^1 \to S^1$ where the mapping f is given by $f(z) = z^n$ for some $n \in Z$.

50. Determine the fundamental group of each of the compact surfaces.

51. Determine the homology with coefficients Z of each of the compact surfaces.

52. Suppose that X is compact and Y is Hausdorff. Suppose that $f: X \to Y$ is continuous, one-to-one and onto. Show that f is a homeomorphism.

53. Suppose that X is compact and Y is Hausdorff. Suppose that $f: X \to Y$ is continuous. Show that f is closed.

54. Show that I and I^2 are not homeomorphic.

55. Show that the Stone-Cech compactification of the integers is not metrizable.

56. Let X be a compact space and suppose that $f: X \to Y$ is a local homeomophism. Show that for each $y \in Y$, $f^{-1}(y)$ is a finite set of points.

57. Suppose that Y is contractible to the point y_0. Let $f: X \to X \times Y$ be defined by $f(x) = (x, y_0)$. Let $p: X \times Y \to X$ be the projection. Show that f and p induce a homotopy equivalence between X and $X \times Y$.

58. Show that for $n, m > 0$, $n \neq m$, R^n is not homeomorphic to R^m.

59. Construct a space X with $H^1(X, Z) \cong Z_n$, the cyclic group of order n.

60. Construct a space X with $H^1(X, Z) \cong Z_n \oplus Z_m$ for any positive integers n and m.

61. Suppose that $X = U \cup V$ with U and V open, simply connected and not disjoint, and that $U \cap V$ is path connected. Show that X is simply connected.

62. Assume that X is connected, locally connected, and locally simply connected. Let $p: \tilde{X} \to X$ be a covering space with covering projection. If $h: X \to X$ is a homeomorphism with fixed point x and $p(\tilde{x}) = x$, show that h lifts to a homeomorphism $\tilde{h}: \tilde{X} \to \tilde{X}$ with $\tilde{h}(\tilde{x}) = \tilde{x}$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{h}} & \tilde{X} \\ \downarrow p & & \downarrow p \\ X & \xrightarrow{h} & X \end{array}$$

if and only if $h_* p_*(\pi_1(\tilde{X}, \tilde{x})) = p_*(\pi_1(\tilde{X}, \tilde{x}))$.

63. Show that there is no retraction of the Möbius band onto its boundary.

64. Suppose that $f, g: X \to S^n$ are two maps such that $f(x)$ and $g(x)$ are not antipodal for all $x \in X$. Show that f and g are homotopic.

65. Let P be a connected finite polyhedron. Show that there is an $\varepsilon > 0$ such that for any maps $f, g: X \to P$, if $d(f,g) < \varepsilon$, then f and g are homotopic.

66. Let $f: S^2 \to T^2$. Show that f is null-homotopic.

67. Let Σ_2 be the dyadic solenoid. Show that Σ_2 is not arcwise connected.

68. Show that any compact metric space can be embedded in the Hilbert cube.

69. Suppose that X is a finite simplicial complex and that A is a subcomplex of X. Define $H_p(X)$ and $H_p(X,A)$. Include definitions of the auxiliary constructions used, such as chains and cycles.

70. Let $T^2 = S^1 \times S^1$ be the 2-dimensional torus. Show that $H_2(T^2) \neq 0$.

71. Let X and Y be simplicial complexes and let $f: X \to Y$ be a continuous map.
(a) Define what it means for g to be a simplicial approximation to f.
(b) State the Simplicial Approximation Theorem.
(c) Prove the Simplicial Approximation Theorem for finite simplicial complexes.

72. State the relationship between $\pi_1(X)$ and $H_1(X)$ for connected finite simplicial complexes. What happens if X is not connected?

73. State the exact homology sequence for a pair of spaces.

74. Prove that a compact Hausdorff space is normal.

75. What is the universal covering space for n-dimensional projective space, P^n?

76. Show that a locally connected connected compact metric space is arcwise connected.

77. Suppose that X is a metric space such that for every map $f: X \to R$, f is bounded. Show that X must be compact.

78. Let X be a finite complex. Show that $\chi(X \times S^1) = 0$.

79. Show that no covering space of the 2-torus has the homotopy type of a figure 8.

80. Show that the complex projective space CP^3 does not have the homotopy type of $S^2 \times S^4$. [Hint: Use the ring structure in cohomology.]

81. Let X be a locally compact Hausdorff space and let $C(X,Y)$ be the space of continuous maps from X to Y with the compact-open topology. Show that the evaluation map

$$e: X \times C(X,Y) \to Y$$

defined by $e(x,f) = f(x)$ is continuous.

82. Let $X = T^2 \vee S^1$, the one-point union of a torus and a circle. Show that if $f: X \to X$ is homotopic to the identity map, then f has a fixed point.

83. State the Jordan Curve Theorem.

84. State the Schoenflies theorem for R^2. Does the corresponding theorem hold for R^3? State the Schoenflies Theorem for R^n for $n > 2$.

85. Compute the homology of the i-skeleton of the n-simplex, Δ^n, where $i \leq n$.

86. Let $n > 1$. Let C be a Hausdorff compactification of R^n. Show that $C \setminus R^n$ is connected.

87. Define retraction and deformation retraction.

88. Show that the fundamental group of a topological group must be abelian.

89. State and prove the Ascoli (Arzela-Ascoli) Theorem.

90. Let V be a complete metric space such that no point is isolated. Can V be a countably infinite set?

91. State the Lefschetz Fixed Point Theorem for compact simplicial complexes.

92. Let A be a countable subset of R^2. Show that $R^2 \setminus A$ is arcwise connected.

93. Compute the homology groups of the nonorientable surfaces of genus one (Klein bottle) and genus two using coefficients (a) Q, (b) Z, and (c) $Z/2$.

94. Give an example of a retraction of a space Y onto a subset X which is not a deformation retraction.

95. Let $0 \to G \xrightarrow{g} H \xrightarrow{h} J \to 0$ be an exact sequence. What does this say about g and h? Prove your claims.

96. What is an exact sequence? What is a short exact sequence?

97. Let $f: S^n \to R^n$ be any continuous function. Show that there is a pair of antipodal points $\{x, -x\} \subset S^n$ such that $f(x) = f(-x)$.

98. Suppose that X and Y are finite complexes and that $p: X \to Y$ is a d-sheeted covering projection. Show that $\chi(X) = d \cdot \chi(Y)$.

99. Suppose that C_* is a free finite chain complex. Show that the Euler characteristic of C_* can be computed from $H_*(C_*)$ as well as from C_*. That is, show that $\sum_{i \geq 0} (-1)^i \cdot \text{rank}(C_i) = \sum_{j \geq 0} (-1)^j \cdot \text{rank}(H_j(C_*))$.

100. Show that the Cantor set less its endpoints is homeomorphic to the irrationals.