Study Guide for Ph.D. Topology Examination

On the examination you will be expected to answer all questions and show all work. Proofs are expected to be detailed enough that there can be no misunderstanding as to their correctness. Explain how counterexamples contradict the statements for which they are counterexamples.

1. Let f be a continuous function from the space X to the circle S^1. Assume that X is connected. Let $e : R \to S^1$ be defined by $e(t) = \exp(2\pi ti)$. Show that f is null-homotopic if and only if there is a mapping $\tilde{f} : X \to R$ such that $e \circ \tilde{f} = f$.

2. Use the Mayer-Vietoris sequence for reduced singular homology to compute $H_*(S^n)$.

3. Use the van Kampen Theorem to compute the fundamental group of the surface of genus two.

4. Show that any map $f : S^2 \to T^2$ is null-homotopic.

5. Let Σ_2 be the dyadic solenoid. Show that Σ_2 is not arcwise connected.

6. Let N be the positive integers with the discrete topology. Let βN be the Stone-Čech compactification. Show that βN is not metrizable.

7. State and prove the Jordan Curve Theorem in the plane.

8. State and prove the Brouwer Fixed Point Theorem.

9. Let $f : S^n \to R^n$ be any continuous function. Show that there is a pair of antipodal points $\{x, -x\} \subset S^n$ such that $f(x) = f(-x)$.

10. Let f, g : $X \to S^n$ be any continuous mappings. Suppose that for all x in X f(x) and g(x) are not antipodal. Show that f and g are homotopic as maps.

11. Let P be a compact polyhedron. Show that there is an $\epsilon > 0$ such that if f, g : $X \to P$ are any two continuous functions such that $d(f(x), g(x)) < \epsilon$ for all $x \in X$, then f and g are homotopic.

12. Show that any compact metric space can be embedded in the Hilbert cube.

13. Let C be the Cantor set. Let Y be any compact metric space. Show that there is a continuous map $f : C \to Y$ such that f is onto.

14. Let X be the set of points in the plane which is the closure of the set $S = \left\{\left(x, \sin\frac{1}{x}\right) \middle| 0 < x \leq 1\right\}$. Show that there is no continuous mapping $f : [0,1] \to X$ which is onto.

15. Let O be an open subset of R^n. Show that O is connected if and only if O is arcwise connected.

16. Suppose that X is a finite simplicial complex. Define $H_p(X)$ and $H_p(X,A)$ carefully, including all definitions.

17. Let $T^2 = S^1 \times S^1$. Show that $H_2(T^2) \neq 0$.

19. Let X and Y be simplicial complexes and $f : X \to Y$ continuous.
(1) Define what it means for g to be a simplicial approximation to f.
(2) State the Simplicial Approximation Theorem.
(3) Prove the Simplicial Approximation Theorem for finite simplicial complexes.

20. State the relationship between $\pi_1(X)$ and $H_1(X)$ for finite simplicial complexes.

21. What is an exact sequence? What is a short exact sequence?

22. State and prove the Five Lemma for short exact sequences.

23. Let $0 \to G \xrightarrow{g} H \xrightarrow{h} J \to 0$ be an exact sequence of groups. What does this say about g and h? Prove.

24. State the Exact Homology Sequence of a Pair.

25. Let $K = K_1 \cup K_2$ be a simplicial complex and $A = K_1 \cap K_2$. State the Mayer-Vietoris Sequence for this. Use it to determine $H_p(S^1 \times D^2)$ where D^2 is a closed disk in R^2.

26. Prove that a compact Hausdorff space is normal.

27. State Urysohn's Lemma. Must the map of the theorem be onto?

28. Let X be a completely regular T_1 space. State what it means for Y to be a Hausdorff compactification for X. Construct two different compactifications of the 2-sphere minus a point.

29. Let $X \subset Y$. What does it mean for X to be a retract of Y. What does it mean for X to be a deformation retract of Y.

30. Let $Y \subset X$. Give an example of a deformation retraction of X onto Y. Give an example of a retraction of X onto Y with Y not a deformation retract of X.

31. Compute $\pi_1(S^1)$. Compute $\pi_1(T^2)$. Compute $\pi_1(S^1 \times D^2)$.

32. Let X be the solid two-holed torus. Use the Seifert-van Kampen Theorem to find $\pi_1(X)$.

33. Define what it means for a space X to be simply connected. Give examples of two simply connected spaces. Show that S^n is simply connected for $n > 1$.

34. What is the universal covering space of the projective plane, P^2?

35. Compute the homology of the i-skeleton of the n-simplex, Δ^n, where $i \leq n$.

36. Show that the fundamental group, $\pi_1(G,e)$, of a topological group G must be Abelian.

37. Compute the fundamental groups of the nonorientable surface of genus one (Klein bottle) and genus two.

38. Compute the homology groups of the Klein bottle and the nonorientable surface of genus two using coefficients (i) Z and (ii) $Z/2$.

39. Show that no covering space of the 2-torus T^2 has the homotopy type of the figure-8, $S^1 \vee S^1$.

40. Show that CP^3 and $S^2 \times S^4$ have different homotopy types. [Hint: Use the ring structure in cohomology.]

41. Suppose that X and Y are finite complexes. Suppose that $p : X \to Y$ is a d-sheeted covering projection. Show that $\chi(X) = d \cdot \chi(Y)$.

42. Suppose that C_* is a free finite chain complex (that is, there is an $N > 0$ such that $C_n = 0$ for $n > N$ and each C_i is a finitely-generated free Abelian group). Show that the Euler characteristic of C_* may be computed from $H_*(C_*)$ as well as from C_*, i.e., show that

$$\sum_{i=0}^{\infty} \operatorname{rank}(C_i)\cdot(-1)^i = \sum_{j=0}^{\infty} \operatorname{rank}(H_j(C_*))\cdot(-1)^j.$$

43. Let $f : S^4 \to S^4$ be a continuous map. Let $f_* : H_4(S^4) \to H_4(S^4)$ be the induced homomorphism. Suppose that f is of non-negative degree. Show that f_* carries a generator g to a multiple mg, where $m \geq 0$.

44. Must a continuous map of S^4 to itself of non-negative degree have a fixed point?

45. Define an action of the infinite cyclic group Z on the punctured plane $X = R^2 \setminus \{(0,0)\}$ by letting the generator g of Z act by $g(x,y) = (2x, \frac{1}{2}y)$. Show that this action defines a group of covering transformations on X, i.e., for each $x \in X$ there is a neighborhood N of x such that only finitely many $z \in Z$ satisfy $zN \cap N \neq \emptyset$.

46. Show that the quotient space in the above problem is not Hausdorff.

47. Let X be a locally compact Hausdorff space and let $C(X,Y)$ be the space of continuous maps from X to Y with the compact-open topology. Show that the evaluation map $e : X \times C(X,Y) \to Y$ is continuous where $e(x,f) = f(x)$.

48. Let X be a finite complex. Show that $\chi(X \times S^1) = 0$.

49. Describe all covering spaces of the torus $T^2 = S^1 \times S^1$.

50. Give an example of a space X for which $\pi_1(X, x_0)$ is not finitely generated.

51. Let A be a closed subspace of a metric space X such that A is homeomorphic to RP^2. Prove that any map $f : A \to T^2$ extends to a map $F : X \to T^2$.

52. Let A be closed in R^n. Prove that each component of $R^n \setminus A$ is path-connected.

53. Let X be a metric space such that every map $f : X \to R$ is bounded. Prove that X is compact.

54. Let $X = T^2 \vee S^1$, the one-point union of a torus and a circle. Show that if $f : X \to X$ is a map homotopic to the identity, then f has a fixed point.

55. Show that $X = T^2 \vee S^1$ does not have the fixed-point property.

56. Let S_g be a closed orientable surface of genus $g \geq 1$. What is the cardinality of $[RP^2, S_g]$, the set of homotopy classes of maps from the projective plane to S_g?

57. Prove that $\pi_1(S^n, x_0) = \{1\}$ if $n \geq 2$.

58. State the Jordan Separation Theorem for R^n.

59. State the Schoenflies Theorem for R^2. Does the corresponding statement hold for R^3? Explain.

60. State Tietze's Extension Theorem for normal spaces.

61. Show that every separable metric space embeds in the Hilbert cube.

62. Show that $R^2 \setminus A$ is path-wise connected for A any countable set. [Hint: Consider the space $C([0,1], R^2)$ with the supremum metric. Let $x \in R^2$ and show that $U_x = ([0,1], R^2 \setminus \{x\})$ is a dense open set in $C([0,1], R^2)$. Apply the Baire Category Theorem.]

63. State and prove the Baire Category Theorem for complete metric spaces.

64. State the Lefschetz Fixed-Point Theorem for compact simplicial complexes.

65. State and prove the Contraction Mapping Theorem for complete metric spaces.