Ph.D. Numercial Analysis

1.

a) Show that for invertible A and a suitably small pertubation E, the exact solutions x and y of $Ax = b$ and $(A+E)y = b$ satisfy

$$\frac{\|y-x\|}{\|x\|} \approx \kappa(A)\frac{\|E\|}{\|A\|}.$$

b) Recall that by backward error analysis of Gaussian elimination with partial pivoting, the computed solution $\hat{x}$ of $Ax = b$ is the exact solution of a nearby system $(A + E)x = b$, where

$$\|E\|_\infty \approx 8n^3\rho\|A\|_\infty\mu.$$

Here μ is the unit roundoff and ρ is the growth factor. Show how to combine the pertubation theory results from part a) with these roundoff error results to obtain results on the accuracy of the computed solution of a linear system via Gaussian elimination with partial pivoting.

2. Write a program, using any language of your choice, that overwrites the upper triangle of a symmetric, positive definite matrix A with its Cholesky factor R using no scratch arrays. It should access only the upper triangle of A. Derive an analytic estimate of the number of flops required.

3. Suppose $A \in \mathbf{R}^{m\times n}$ has full column rank and $b \in \mathbf{R}^m$. Show that x is a solution of $A^TAx = A^Tb$ iff it minimizes $\|b - Ax\|_2$.

4. Show how the singular values of $A \in \mathbf{R}^{m\times n}$ relate to the eigenvalues of A^TA.

5. Show that if $A \in \mathbf{R}^{m\times n}$, then $\|A\|_2 = \sigma_1$, the largest singular value of A.

6. Carefully state the relation between the SVD of a matrix and its 2-norm distance from matrices of lower rank.

7. Let $x = [7; 24]$. Determine matrices M, H, and G such that Mx, Hx, and Gx have zero second coordinate, where

a) M is a Gauss transformation,

b) H is a Householder transformation, and

c) G is a Givens rotation.

8. Let f be a n times continuously differentiable function. Let $L_n(f, x)$ be the unique Lagrange interpolation polynomial of degree $\leq n-1$ such that

$$L_n(f, x_i) = f(x_i), i = 1, 2, ..., n.$$

Prove that for all x in the interval $[a, b]$, we have

$$f(x) - L_n(f, x) = \frac{\omega_n(x)}{n!} f(\eta)$$

where η lies on $[a, b]$, $\omega_n(x) = (x - x_1)...(x - x_n)$, and $a < x_n < x_{n-1} ... < x_1 < b$.

9. Given the value of f and its derivative at $x = 0$, and the value of f at $x = 1$, use Lagrange interpolation polynomials to obtain an estimate for the integral of f over the interval $[1, 2]$. Obtain an estimate for the derivative of f at $x = 2$.

10. State Euler's method for approximating the solution to the differential equation

$$\frac{dx}{dt} = f(x(t)), \quad x(0) = x_0.$$

Show that the error in Euler's method is $O(\Delta t)$ for t sufficiently small, where Δt is the step size.