PH.D. QUALIFYING EXAM IN COMPLEX ANALYSIS

Give complete proofs and computations. Partial credit will be given where justified.

1) Evaluate the integral

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\, dx .$$

2) Find the order of each of the following entire functions:

(a) $\frac{\sin\sqrt{z}}{\sqrt{z}}$

(b) $\Pi_{k=1}^{\infty}(1+\frac{z^k}{k!})$

(c) $\Pi_{k=1}^{\infty}(1+\frac{z^k}{k\ln^2 k})$

3) Let $G \subset \mathbb{C}$ be a region and $\{f_n\} \subset H(G)$ be a sequence of injective functions which converges to f in $H(G)$. Prove that either f is also injective or it is constant on G.

4) Let f be an analytic function mapping the unit disc D into itself and having two or more distinct fixed points in D. Show that f must be the identity function $f(z) = z$ for all $z \in D$.

5) Construct the analytic function which map the unit disc D conformally onto the angular region $|\arg(z)| < a$, for fixed $a \in (0, \pi)$, and satisfies $f(0) = 1$. With the help of this function, prove that if g is a function which is analytic in D and satisfies both $g(0) = 1$ and $|\arg(g(z))| < a$ $(a \in (0, \pi))$, then $|g'(0)| \leq 4a/\pi$.

6) Let f be an entire function of finite order. Prove that if the order is not an integer, then f must have infinitely many zeros. Does there exist an entire function of infinite order with finitely many zeros? Explain.

7) Let n be a natural number and a be a real number such that $a > e$. Show that the equation $e^z - az^n = 0$ has exactly n solutions inside the unit disc.

8) Suppose f is entire and that there exists a bounded sequence $\{a_k\}_{k\in\mathbb{N}}$ of distinct real numbers such that $f(a_k)$ is real for every k. Prove that $f(x)$ is real for all real x.