PhD Complex Analysis Exam, January 3, 2014.

Do any 9 problems.

The notation $f \in H(\Omega)$ means f is holomorphic in the domain Ω.

1. Prove the fundamental theorem of algebra.

2. Let Ω be a domain on the complex plane and M be a finite positive constant. Suppose F is a family of holomorphic functions on Ω. If $|f(z)| \leq M$ for all $z \in \Omega$ and all $f \in F$, prove that F is equi-continuous on Ω.

3. Let $f_n \in H(\Omega), n = 1, 2, 3, ...$ be one to one in Ω. If $f_n \to f$ uniformly on compact subsets in Ω, prove that f is either one to one or constant. Show with examples that both conclusions can occur.

4. Let $f \in H(\Omega)$. If f has no zeros in Ω, prove that $\ln |f|$ is harmonic in Ω.

5. Evaluate the integral

$$\int_0^\infty \frac{\ln x}{(x^2 + b^2)^2}, b > 0.$$

6. Let $\mathbb{C}$ be the complex plane. Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous and f is holomorphic except in $[-1, 1]$. Prove that f is entire.

7. Prove that a doubly periodic entire function is constant.

8. Let P be a polynomial and C the circle $|z - a| = R$, Evaluate the integral $\int_C P(z) d\bar{z}$.

9. Let $f, F \in H(U)$, where $U = \{z : |z| < 1\}$ and let $R_f = \{f(z) : z \in U\}$ and $R_F = \{F(z) : z \in U\}$. Suppose F is one to one in U, $f(0) = F(0)$ and D_f is contained in D_F. Prove that there exists an $\omega \in H(U)$ such that $f(z) = F(\omega(z))$ and $|\omega(z)| \leq |z|$. Also show that the equality holds if and only if $D_f = D_F$.

10. Suppose $f \in H(\Pi^+)$, where Π^+ is the upper half plane and $|f| \leq 1$. How large can $|f'(i)|$ be? Find the extremal functions.

11. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.