PhD Complex Analysis Examination. September 2011. Do all 9 problems.

The symbol U always denotes the unit disc: $|z| < 1$; $\mathbb{C}$, the complex plane; Ω, an open region in $\mathbb{C}$; and $\Re z$, the real part of z.

1. Evaluate the integral $\int_0^\infty \frac{(\log x)^3}{1+x^2} dx$.

2. Let $f_n, n = 1, 2, 3, ...$, be a sequence of one-to-one analytic functions on Ω. If f_n converges uniformly to f on any compact subset of Ω, prove that f is either constant or one-to-one. Show by examples that both conclusions can occur.

3. Let F be a family of analytic functions f on U such that $\Re f > 0$ and $f(0) = 1$. Prove that F is a normal family. Can the condition $f(0) = 1$ be omitted? Justify your answer.

4. Suppose f is a conformal mapping of U onto a square with center at 0 and $f(0) = 0$. Prove that $f(iz) = if(z)$. If $f(z) = \sum_0^\infty c_n z^n$, prove that $c_n = 0$ unless $n - 1$ is a multiple of 4.

5. Let R be a rational function such that $|R(z)| = 1$ if $|z| = 1$. Prove that R is of the form $R(z) = cz^m \prod_{n=1}^k \frac{z-a_n}{1-z\overline{a_n}}$,where c is a constant with $|c| = 1$, m is an integer and $a_1, a_2, ..., a_k$ are complex numbers such that $a_n \neq 0$ and $|a_n| \neq 1$.

6. Let f be analytic in Ω. If f has no zeros in Ω, prove that $\log |f|$ is harmonic in Ω.

7. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a continuous function and f is analytic everywhere except possibly on $[-1, 1]$. Prove that f is an entire function.

8. Suppose f and g are both analytic in U, and neither of them has a zero in U. If $\frac{f'}{f}(1/n) = \frac{g'}{g}(1/n)$ for $n = 2, 3, 4, ...$, find a simple relation between f and g.

9. Construct a conformal map $w = f(z)$ which maps U onto the angular region A : $|\arg(w)| < \alpha$ and $f(0) = 1$, where $0 < \alpha < \pi$.