Qualifying Exam in Complex Analysis – September, 2009

1. A *fixed point* of a function f is a point x with $f(x) = x$. Let G be an open, simply connected subset of $\mathbb{C}$. If f is analytic on G, $f(G) \subset G$, and f has two fixed points, show that f is the identity map.

2. Let G be an open, connected subset of $\mathbb{C}$ and f be analytic on G. Note that we are not assuming that G is simply connected.

(a) Assume now that for another open, connected set H, $f(G) \subset H$ and $h : H \to \mathbb{R}$ is harmonic. Show that $h \circ f$ is harmonic.

(b) Prove or disprove: Both the real and imaginary parts of f are harmonic functions.

(c) Prove or disprove, if $u : G \to \mathbb{R}$ is harmonic, then there is a harmonic function $v : G \to \mathbb{R}$ so that $u + iv$ is analytic.

3. Give an explict biholomorphism from the "wedge" $\{z : 0 < \arg(z) < \pi/2\}$ onto the open unit disk.

4. Evaluate the integral, and justify each step.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\, dx.$$

5. Let f be an entire function of finite order. Prove that if the order is not an integer, then f must have infinitely many zeros. Does there exist an entire function of infinite order with finitely many zeros? Explain.

6. Assume each g_n is entire and $g_n \to g$ uniformly on compact sets in $\mathbb{C}$ (i.e. $g_n \to g$ in $H(\mathbb{C})$). If each g_n has only real zeros, show that g has only real zeroes.

7. Assume that f and g are analytic and nonvanishing on $\{z \in \mathbb{C} : |z| < 2\}$ and that $|f(z)| = |g(z)|$ when $|z| = 1$. Show that there is a constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $f(z) = \lambda g(z)$ for all $z \in B_2(0)$.

8. Assume that f is entire, and for some integer $n > 0$,

$$\lim_{n\to\infty} \frac{f(z)}{z^n} = c,$$

for some finite, nonzero complex number c. What can you say about f?