Ph.D. Qualifying Exam in Complex Analysis

Give complete proofs and computations. Partial credit will be given where justified. In the following, $\mathbb{C}$ denotes the set of complex numbers and $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

1) Assume that f is an entire function with $f(0) = 0$. Consider the family of functions $\{f_n\}$ where f_n is the n^{th} iterate of f, i.e. $f_n = f \circ f \circ \cdots \circ f$ (n times).

(a) If $|f'(0)| < 1$, show there is a open set U containing zero so that the family $\{f_n\}$ is normal when restricted to U.

(b) If $|f'(0)| > 1$, show there is no open set U containing zero so that the family $\{f_n\}$ is normal when restricted to U.

2) Let G and G' be open, simply connected proper subsets of $\mathbb{C}$. Show that for any pair of points $p \in G$ and $p' \in G'$ there is a bijective, analytic function $f : G \to G'$ with $f(p) = p'$.

3) (a) Show that

$$\prod_{n=2}^{\infty}(1 - \frac{1}{n^2}) = \frac{1}{2}\,.$$

(b) For which values of z does the following product converge?

$$\prod_{n=0}^{\infty}(1 + z^{3n})$$

4) Find an explicit biholomorphism from the semidisk $E := \{z \mid |z| < 1, \mathrm{Re}(z) > 0\}$ onto the unit disk D.

5) Let $0 < a < 1$. Evaluate the integral

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\, dx\,.$$

6) Let f be an entire function of finite order. Prove that if the order is not an integer, then f must have infinitely many zeros. Does there exist an entire function of infinite order with finitely many zeros? Explain.

7) Let f be entire and assume that there exist $M, R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leq M|z|^n$ for all $z \in \mathbb{C} \setminus B(0; R)$. Prove that f is a polynomial of degree less than or equal to n.

8) Let f have a power series expansion about 0 with radius of convergence 1. Assume, in addition, that all of the coefficients of this power series expansion are greater than or equal to zero. Prove that 1 must be a singular point for f.