Qualifying Exam in Complex Analysis   August, 2007

1. Let $\overline{D}$ be the closure of the unit disk. Assume $f : \overline{D} \to \mathbb{C}$ is continuous with f analytic on D, $|f(z)| > 3$ for $|z| = 1$, and $f(0) = 1 - 2i$. Must f have a zero in the unit disk? Prove your answer is correct.

2. Let G be a simply connected region. A function $f : G \to G$ is said to be biholomorphic if f is bijective, analytic, and f^{-1} is also analytic. If z_1, z_2 are distinct elements of G and $f_1, f_2 : G \to G$ are biholomorphisms with $f_1(z_1) = f_2(z_1)$ and $f_1(z_2) = f_2(z_2)$, prove that $f_1 = f_2$.

3. Assume that f is entire, $f(0) = 3 + 4i$, and $|f(z)| \leq 5$ when $|z| < 1$. What is $f'(0)$?

4. Let $\mathcal{F}$ be the collection of analytic mappings of open unit disk D into $\{Re(z) > 0\}$ such that $f(0) = 1$. Show that $\mathcal{F}$ is a normal family.

5. Let h_n be a sequence of harmonic functions on the connected open set G and assume that $h_n \to h$ uniformly on compact subsets of G. Show that h is harmonic.

6. Assume that f and g are entire and for all $z \in \mathbb{C}$, $|f(z)| \leq |g(z)|$. Show that for some constant c with $|c| \leq 1$, $f = cg$.

7. Assume that f is a meromorphic function on $\mathbb{C}$. A complex number w is called a *period* of f if $f(z + w) = f(z)$ for all $z \in \mathbb{C}$.

(a) If w_1 and w_2 are periods of f, show that for all integers n_1 and n_2, $n_1 w_1 + n_2 w_2$ is also a period of f.

(b) If G is a bounded region of $\mathbb{C}$, show that f has at most finitely many periods in G.

8. Let p be the polynomial $p(z) = a_0 + a_1 z + \ldots a_n z^n$. Show that for each $j = 0, 1, \ldots, n$,

$$|a_j| \leq \max\{|p(z)| : |z| = 1\}. \tag{1}$$