Qualifying Exam in Complex Analysis May, 2007

1. Assume that $h : \mathbb{C} \to \mathbb{R}$ is harmonic, and further that $h(z) > 0$ for all $z \in \mathbb{C}$. Show that h is constant.

2. Prove the Fundamental Theorem of Algebra: if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ with each coeffcient $a_j \in \mathbb{C}$, then there exists a $w \in \mathbb{C}$ with $p(w) = 0$.

3. Recall that $H(G)$ is the set of all holomorphic functions defined on the open, connected set G and that $f_n \to f$ in $H(G)$ exactly when $f_n \to f$ uniformly on all compact subsets of G. Assume that the family $\{f_n\} \subset H(G)$ is locally bounded and further that $f \in H(G)$ has the property that the set

$$\{z \in G : \lim_{n\to\infty} f_n(z) = f(z)\}$$

has a limit point in G. Show that $f_n \to f$ in $H(G)$.

4. A *fixed point* of a function f is a point x with $f(x) = x$.
Let $D = \{z \in \mathbb{C} : |z| < 1\}$. If $f : D \to D$ is analytic and has two fixed points, show that f is the identity map.

5. If $a_0 \geq a_1 \geq \cdots \geq a_n > 0$, show that $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has no roots in $|z| < 1$.

6. For which z does the infinite product

$$\prod_{n=0}^{\infty} (1 + z^{2^n})$$

converge?

7. Let G be an open, connected and *bounded* subset of $\mathbb{C}$. Assume that f is continuous on the closure $Cl(G)$ of G, analytic on G, and further that $|f(z)| = 1$ for all $z \in Fr(G)$ where $Fr(G)$ is the topological frontier of G. Show that either f has a zero in G or else there is a constant c with $|c| = 1$ and $f(z) = c$ for all $z \in G$.

8. Evaluate the integral, and justify each step.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2} \, dx.$$