Sept. 2006

PH.D. QUALIFYING EXAM IN COMPLEX ANALYSIS

Give complete proofs and computations. Partial credit will be given where justified. In the following, $\mathbb{C}$ denotes the set of complex numbers and $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

1) Evaluate the integral

$$\int_0^\infty \frac{\log x}{(1+x^2)^2}\, dx .$$

2) (a) Let f be analytic at $z \in \mathbb{C}$. Prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2 .$$

(b) Let $f_1, f_2, \dots, f_n$ be analytic in the region G. Prove that $|f_1|^2 + |f_2|^2 + \dots + |f_n|^2$ is harmonic on G if and only if f_k is constant, for all $k = 1, \dots, n$.

3) Let $\{f_n\}$ be a sequence of functions, each analytic in the open set G, which converges to f uniformly on all compact subsets of G. Prove that f is analytic in G.

4) Let $P(z) = z^7 + z^5 + 5z^3 + 1$. Find the number of zeros of P counted according to their multiplicities in
(a) $\{z \in \mathbb{C} \mid |z| < 1\}$
(b) $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$
(a) $\{z \in \mathbb{C} \mid |z| > 0\}$

5) Let G be a region and f be analytic in G. Prove that if $f(G)$ is a subset of a circle, then f is a constant function.

6) Suppose f is a nonconstant analytic function in D such that $|f(z)| \le 1$. Let $a = f(0)$. By considering the function

$$g(z) = \frac{f(z) - a}{1 - \bar{a} f(z)},$$

prove that

$$\frac{|a| - |z|}{1 + |a||z|} \le |f(z)| \le \frac{|a| + |z|}{1 - |a||z|}$$

for $|z| < 1$.

7) (a) Does there exist an analytic mapping $f : D \to D$ such that $f(0) = 0$ and $f(\frac{i}{4}) = \frac{1}{3}$? Justify your answer.

(b) Does there exist an analytic mapping f of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ into itself such that $f(3) = 3$ and $f(9) = 6$? Justify your answer.

8) Let $\mathcal{F}$ be the collection of all analytic mappings of D into $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ such that $f(0) = 1$. Show that $\mathcal{F}$ is a normal family.