May 2006

Ph.D. Qualifying Exam in Complex Analysis

Give complete proofs and computations. Partial credit will be given where justified.

1) Let $0 < a < 1$. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\, dx,$$

where $a \in (0,1)$.

2) (a) Prove that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

(b) Deduce from (a) that

$$\tan \pi z = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{z}{(2n-1)^2 - 4z^2}.$$

3) Let G be a region and $u : G \to \mathbb{R}$ be harmonic. Let v be a harmonic conjugate to u in G. Show that the product $u \cdot v$ is harmonic.

4) (a) Show that if f is analytic on the closed disk $\overline{B(0;1)}$, then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^{R} |f(a + re^{i\theta})|^2 r\, dr d\theta.$$

(b) Let G be a region and $M \in \mathbb{R}$ be fixed. Let $\mathcal{F} = \{f \in H(G) \mid \int_G |f(z)|^2\, dxdy \leq M\}$. Show that $\mathcal{F}$ is a normal family.

5) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic on $\overline{B(0;1)}$. Let

$$F(z) = \frac{1}{2\pi i} \int_{\partial B(0;1)} \frac{f(w)}{w} g\left(\frac{z}{w}\right) dw, \ z \in B(0;1),$$

with the contour taken with counterclockwise orientation. Show that

$$F(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \ z \in B(0;1).$$

6) Let $G = \{z \in \mathbb{C} \mid |z| < 1 \text{ and } |2z - 1| > 1\}$, and let $f \in H(G)$.

(a) Must there exist a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of G? Explain your reasoning.

(b) Must there exist such a sequence which converges to f uniformly on G? Explain your reasoning.

(c) Must there exist such a sequence which converges to f uniformly on G, if f is now required to be analytic in some neighborhood of $\overline{G}$? Explain your reasoning.

7) Let G be a region and A be a discrete and relatively closed subset of G. Assume that $f \in H(G \setminus A)$ is injective.

(a) Prove that no point of A can be an essential singularity of f.

(b) Prove that if $a \in A$ is a pole for f, then a is a pole of order 1.

8) Let f be analytic on $\overline{B(0;1)}$. If $|f(z)| < 1$ on $\partial B(0;1)$, prove that there exists a unique $z_0 \in B(0;1)$ (counting multiplicity) such that $f(z_0) = z_0$. If $|f(z)| \leq 1$ on $\partial B(0;1)$, does the same conclusion hold? Justifiy your assertion.