25 May, 2000 **Complex Analysis Ph.D Exam** Profs King Shen Walsh

Notes. Please write up solutions to the eight problems $C1$–$C8$, below. Please write LARGE, as the grader's eyes are older and weaker than your eyes ...

A ***square*** $S \subset \mathbb{C}$ is a set of the form $[a, a+L] \times [b, b+L]$, where L is positive. Use "$s :=$ Foo" to mean that Foo is the *definition* of the new symbol s.

$C1$: Find complex numbers a, b, c, d, with $ad-bc \neq 0$, so that the Möbius transformation

$$\mu(z) := \frac{az+b}{cz+d}$$

carries the imaginary axis to the circle whose radius is 2 and whose center is $3 = 3 + 0i$.

$C2$: With B the open unit ball $|z| < 1$, consider a non-constant analytic function $h: B \to \mathbb{C}$.

i: Suppose that $\Re(h(z)) \geqslant 0$ for each $z \in B$. Prove that the inequality can then be strengthened to "$>$".

ii: With $\Re(h(z)) > 0$ on B, suppose further that $h(0) = 1$. Prove, for each $z \in B$, that

$$\frac{1-|z|}{1+|z|} \leqslant \left|h(z)\right| \leqslant \frac{1+|z|}{1-|z|}.$$

$C3$: Suppose that $P()$ is a monic polynomial with degree $N \geqslant 1$. Let $\alpha_1, \ldots, \alpha_N$ be an enumeration (with multiplicity) of the zeros of $P()$.

①: Suppose that $\forall k : \Re(\alpha_k) > 0$. Prove that *all* the zeros of the *derivative*, P', also lie in the positive half-plane, as follows: Establish that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + \cdots + \frac{1}{z-\alpha_N},$$

then use it to complete the proof.

②: Prove *Lucas's theorem*: *If all the zeros of a non-constant polynomial P lie in a convex polygon $Q \subset \mathbb{C}$, then all the zeros of P' lie in Q.*

③: Show that (②) can *fail* if $P()$ is allowed to be a rational function: Namely, by letting

$$P(z) := \frac{z}{z^2+1},$$

find a half-plane H which owns a zero of P' but has no zero of P.

$C4$: Use the Residue Calculus to compute

$$I := \int_0^{+\infty} \frac{1}{[x^4+4] \cdot [x^2+9]^9}\, dx$$

To save arithmetic, you may define some **explicit** points $P_1, \ldots, P_L \in \mathbb{C}$ (what should L be?) and explicit functions $h_1, \ldots, h_L$, and then may express your answer in the form

$$I = \Big[h_1(P_1) + \cdots + h_L(P_L)\Big] \cdot \text{Constant}.$$

(Do not bother to perform the function-evaluation.)

$C5$: a: State (but do not prove) *Morera's Theorem.* (You may use this without proof in (b), if you wish.)

b: Prove this version of the *Schwarz Reflection Principle*: *Suppose f is continuous in the closed upper half-plane $H := \mathbb{R} \times [0, \infty)$ and is analytic on the interior of H. Further suppose that f is real-valued on the real-axis. By defining $\Phi := f$ on H, and*

$$\Phi(z) := \overline{f(\overline{z})}, \quad \text{for all } z \in \mathbb{C} \setminus H,$$

extend f to all of $\mathbb{C}$. Then this Φ is analytic.

$C6$: ❶: Please state *Picard's Theorem.*

❷: Let f be meromorphic in the whole complex plane. Suppose that the range of f omits three distinct values (one of them can be ∞). Prove that f is constant.

❸: Suppose that f and g are entire functions such that, on $\mathbb{C}$,

$$f^3 + g^3 = 1.$$

Prove that f and g are each constant functions. [*Note:* Symbol f^3 means $f \cdot f \cdot f$.]

$C7$: Fix a real $b > 0$. Write down an entire function, f, that vanishes *precisely* on the sequence $(z_n)_{n=1}^{\infty}$, where $z_n := n^b$.

$C8$: Show that all roots of polynomial $P(z) := z^5 + 15z + 1$ lie in the ball $|z| < 2$, but that only one root satisfies $|z| < \frac{3}{2}$.

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