Ph.D. Complex Analysis Exam

June 2, 1997

Do each of the problems.

1. Evaluate the integral $\int_0^\infty \frac{x^{a-1}}{1+x} ax$, $0 < a < 1$.

2. Suppose $\Omega = \{z = -1 < Re\, z < 1\}$. Find an explicit formula for the $1-1$ conformal mapping f of Ω onto $\{z : |z| < 1\}$ for which $f(0) = 0$ and $f'(0) > 0$. Compute $f'(0)$.

3. Let $\mathcal{F}$ be the class of all analytic functions f in $\mathbb{D} = \{z : |z| < 1\}$ for which

$$\iint_{\mathbb{D}} |f(z)|^2 axdy \le 1.$$

Is this a normal family? Justify your answer.

4. Let g be a polynomial. Show that

$$2\iint_{\mathbb{D}} (1-|z|^2)|g'(z)|^2 axdy \le \int_0^{2\pi} |g(e^{i\theta})|^2 d\theta, \text{ where } \mathbb{D} = \{z : |z| < 1\}$$

5. (a) Give a definition of subharmonic functions.
   (b) If f is analytic in Ω, prove that $\ln(1 + |f(z)|^2)$ is subharmonic.
   (c) Let f be analytic in Ω. Is $\ln|f(z)|$ subharmonic in Ω? Justify your answer.

6. Construct an entire function whose zero set is the set of Gaussian integers $a + bi$, where $a, b \in \mathbb{Z}$.

7. For $|z| < 1$, define $f(z)$ by

$$f(z) = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots$$

Extend $f(z)$ analytically in a suitable way so that it can be defined at $z = i$. Using your analytic continuation, determine the radius of convergence of the Taylor series of f with $z = i$ as center.

8. Show $ze^z = 1$ has infinitely many solutions.

9. Define the order and genus of an entire function. What is the relation between the order and genus? Find the order of $\sin z$.