1. Show, if $a > e$, then $e^z - az^n$ has n zeros in $|z| < 1$.

2. Determine all entire functions f such that $|f| = 1$ for $|z| = 1$.

3. Let G denote the punctured disc $0 < |z| < \delta$ and let $f = z\cos(\frac{1}{z})$. Determine the set $f(G)$.

4. Let f be an entire function with a finite number of zeros. Show if for each $c > 0$

$$\lim_{z \to \infty} \frac{|f(z)|}{e^{c|z|}} = 0,$$

then f is a polynomial.

5. Let $a_n \geq 0$ and $f = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $\infty > R > 0$. Show that $z = R$ is a singularity of f. (Suggestion: Show, if $R > r > 0$ and

$$\sum \frac{f^{(n)}(r)}{n!}(z - r)^n$$

has radius of convergence R', then for each $|z_0| = r$

$$\sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

has radius of convergence at least R'. Conclude f is analytic in $|z| < R' + r$.)

6. Let

$$f(z) = 1 + \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{5^z} + \frac{1}{7^z} + \frac{1}{8^z} + \dots.$$

Show that f is analytic in $Re z > 1$. Analytically continue f beyond this half plane. Is f analytic at $z = 1$?

7a. Show if u is harmonic in a region G then

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

is analytic in G.

b. Show if u is harmonic on the annulus $G = \{z : r < |z| < R\}$, then there exists an analytic function f on G and a constant c such that

$$u = Re f + c\log(|z|).$$