Present the solutions with the necessary details, so the partial credit can be properly assessed. Do all 9 problems. The problems are 10 points each.

1. Construct a conformal map which maps the vertical strip $0 < \mathrm{Re}\, z < 1$ onto the upper half plane.

2. The function
$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$
is analytic in the unit circle $|z| < 1$. Can it be continued analytically beyond the circle $|z| = 1$. Discuss the nature of the singularities (if any) on $|z| = 1$.

3. (a) Show that the function
$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$
is analytic in $\mathrm{Re}\, z > 0$

(b) Show that it can be analytically continued to the entire complex plane.

4. Let $f(z)$ be analytic in the disc $|z| \le 1$. If

$$|f(z)| \le 1 \quad \text{for } |z| \le 1,$$

prove that

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \quad \text{for } |z| \le 1.$$

5. Let $F(z)$ be an entire function with the property that $\operatorname{Re} F(z)$ is bounded from above. Prove that $F(z)$ is constant.

6. Let Ω be an open connected subset of $\mathbb{C}$. Let $\{f_n\}$ be a sequence of one to one analytic functions on Ω and assume that $\{f_n\}$ converges uniformly on any compact subset of Ω to a function f. Show that f is either one to one or a constant, and show that both possibilities can occur.

7. Suppose that f is entire, of order ρ. Let $n(r)$ be the number of zeros (counting the multiplicity) of f in the disk $\{z: |z| \leq r\}$. Prove that

$$\limsup_{r \to \infty} \frac{\log n(r)}{\log r} \leq \rho.$$

And show, by examples, that the equality can happen.

8. The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left\{\frac{f''}{f'}(z)\right\}^2$$

is called the Schwarzian derivative of f. Prove the following:

ⓐ If $f(z) = a(z - z_0)^m + \cdots$, where m is an integer,

then $\{f, z\} = A\,(z - z_0)^{-2} + \cdots$

ⓑ Find the value of A

ⓒ What can you say about $\{f, z\}$ at the point z_0 if $m = \pm 1$?

9 (a) Evaluate

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^z}{z(z+1)}\,dz \qquad \text{for } x>1$$

Hint: Write the above integral as $\lim_{t\to\infty} I(t)$, where

$$I(t) = \frac{1}{2\pi i}\int_{2-it}^{2+it},$$

and evaluate $I(t)$ by using the residue theorem on the rectangle $2-it$, $2+it$, $b+it$ and $b-it$ where b is a large negative number.

(b) What happen if $0<x<1$?