Complex Analysis Ph.D. Examination

May 20, 1992

Do all problems and write each problem on a separate sheet. Show all the necessary details so partial credit can be properly assessed.

1. Construct a conformal map which will take the intersection of the disks $|z| < 1$ and $|z-1| < 1$ to the unit disk.

2. Evaluate $\int_0^\infty \frac{\log x}{1+x^4}dx$ using contour integration.

3. Let $f(z)$ be entire of order 1 and the zero set of f, $Z(f) = \{0, \pm 1, \pm 2, \pm 3, ...\}$. Suppose $f'(0) = 1$, $f(\frac{1}{2}) = 1$ and f is bounded on the imaginary axis. Show that $f(z) = \sin \pi z$.

4. Let $P_n(z) = \frac{1}{n}\frac{d^n z^n(1-z)^n}{dz^n}$ be the n^{th}-Legendre polynomial (of degree n). Show that $P_n(-1) \leq (1+\sqrt{2})^{2n}$. (Hint: Write $P_n(-1) = \frac{1}{2\pi i}\int_c \frac{\zeta^n(1-\zeta)^n}{(\zeta+1)^{n+1}}d\zeta$ over a suitable circle centered at -1).

5. (a) Let f be entire and $f(z) \longrightarrow \infty$ as $z \to \infty$. Show that f is a polynomial.

   (b) If in part (a) we replace f entire by f analytic in $\mathbf{C} - \{0\}$, then what can be said?

6. Let $f(z)$ be analytic in a domain D and on its boundary C. If $|f(z)|$ is constant on C, show that unless $f(z)$ reduces to a constant there must be at least one zero of $f(z)$ in D.

7. Show that the sum of the series

$$\frac{1}{1-z} - \frac{z}{1-z^2} - \frac{z^2}{1-z^4} - \frac{z^4}{1-z^8} - \frac{z^8}{1-z^{16}} - \cdots$$

is 1 when $|z| < 1$, but is 0 when $|z| > 1$. This seems to contradict analytic continuation. How is this possible?

8. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| < \dfrac{1}{1-|z|}$ $(|z| < 1)$, show that the coefficients a_n in the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

are subject to the inequality

$$|a_n| \leq (n+1)(1+\frac{1}{n})^n < e(n+1)$$

9. (a) Show that the function

$$w = \left(\frac{1+z}{1-z}\right)^{\frac{2\alpha}{\pi}}, \quad 0 < \alpha < \pi,$$

maps $|z| < 1$ onto the angular sector $|\arg\{w\}| < \alpha$.

(b) Prove that if $f(z)$ is analytic in $|z| < 1$ and satisfies $f(0) = 1$ and $|\arg\{f(z)\}| < \alpha$, then $|f'(0)| < \dfrac{4\alpha}{\pi}$.