Complex analysis Ph.D. Examination May 17, 1990

1. Show that $u(x,y) = x^3 - 3xy^2 + x$ is harmonic in R^2. Find a harmonic conjugate for u.

2. Let G be the region which is the intersection of the disks $|z| < 1$ and $|z-1| < 1$. Find a function which maps G conformally onto the unit disk.

3. Evaluate $\int_0^\infty \cos x / (1+x^4)^2 \, dx$ using contour integration.

4. Show that $f(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s$ is analytic in $\operatorname{Re} s = \sigma > 1$. Prove that f can be analytically continued to all of C as an entire function. (Hint: Write f(s) in terms of $\zeta(s)$ in $\sigma > 1$. Use properties of $\zeta(s)$ to show that f can be made entire.)

5. Let $f(z) = \sum_{n=1}^{\infty} z^n/n^2$. What is the radius of convergence R of this series? Is the series convergent at all points $|z| = R$? At what points of $|z| = R$ is f analytic? (Hint: Think of f as the primitive of $-\log(1-z)/z$.)

6. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

7. We say that z_0 is a fixed point of f if $f(z_0) = z_0$. Suppose f is analytic in a region D containing the closed unit disk.

   (a) How many fixed points must f have in this disk?

   (b) If $|f(z)| > 2$ when $|z| = 1$ and $f(0) = 1$, must f have a zero in this unit disk? Justify your answers in both parts.

8. Suppose f is an entire function and $\operatorname{Re} f(z) \leqq A + B|z|^k$ for all z, where A,B and k are positive numbers. Prove that f must be a polynomial. Describe the polynomial as detail as possible when $A = 0$.

9. Suppose that f is a conformal map of unit disk onto a square with center at 0 and $f(0) = 0$. Prove that $f(iz) = if(z)$. If $f(z) = \sum^{\infty} c_n z^n$, prove that $c_n = 0$ unless $n-1$ is a multiple of 4. (Hint: Show that if(z) is also a conformal map of the unit disk onto the same square.)

10. Deduce from the identity $\sin \pi z = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$ that

$$\sum_{n=1}^{\infty} (n^2 + k^2)^{-1} = (\pi/2k)(e^{2\pi k} + 1)/(e^{2\pi k} - 1) - 1/(2k^2).$$