COMPLEX ANALYSIS EXAMINATION

Present your work clearly and do not leave any gaps in the proofs. Do all 9 problems.

1. Suppose f is one to one and entire. Show that $f(z) = az + b$, $a \neq 0$.

2. Let $J \neq 0$ be a complex number. Prove that if the series $\sum_{n=0}^{\infty} a_n J^n$ converges, then the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < |J|$.

3. Evaluate

$$\int_0^{\infty} \frac{\cos x}{1 + x^2}\, dx.$$

4. a) Let $f(z) = \sum_{n=0}^{\infty} z^{2^n}$, $|z| < 1$. Show that the unit circle is a natural boundary for f.

b) Let $g(z) = \prod_{n=0}^{\infty} (1 + z^{2^n})$. Discuss the behavior of $g(z)$ on $|z| = 1$.

5. Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$

6. Suppose f is analytic in the upper half plane $\operatorname{Im} z > 0$ and $|f| \leq 1$. How large can $|f'(i)|$ be ? Find the extremal functions.

7. Suppose $\{f_n\}$ is a sequence of analytic functions on the region D, none of the functions f_n has a zero in D, and $\{f_n\}$ converges to f uniformly on compact subsets of D. Prove that either f has no zero in D or $f(z) = 0$ for all z in D.

8. a) Suppose f is analytic in a region D containing the closed unit disc $|z| \leq 1$, $|f(z)| > 2$ if $|z| = 1$ and $f(0) = 1$. Must f have a zero in the unit disc?

b) Suppose f and g are entire functions, and $|f(z)| \leq |g(z)|$ for every z. What conclusion can you draw?

9. Suppose f is an entire function and $f(0) = 1$, let $M(r) = \sup_{0 \leq t \leq 2\pi} |f(re^{it})|$ and $n(r)$ be the number of zeros of f in the disc $|z| \leq r$. Prove that

$n(r) \ln 2 \leq \ln M(2r)$.