Ph.D. exam on Measure and Integration Theory

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I. State the following theorems:

1. The monotone convergence theorem for nets in L^1
2. The Hahn decomposition theorem
3. The Egorov theorem
4. The Vitali convergence theorem
5. The Radon Nikodym theorem
6. The Fubini theorem
7. The representation of the dual of L^p.
8. The Lebesgue dominated convergence theorem

II. Prove one of the theorems 6 or 7.

III. Solve 5 of the following problems

In what follows, (X, Σ, μ) is a measure space.

1.) Let (f_n) be a sequence of real valued ~~fu~~ measurable functions on X and f a real measurable function on X. Prove that (f_n) converges to f in μ-measure ($f_n \xrightarrow{\mu} f$) iff every subsequence (f_{n_i}) contains a further subsequence $(f_{n_{i_j}})$ converging to f, μ-a.e.

2.) Let (f_n) be a sequence of positive, μ-integrable functions such that $\sum_n \int f_n \, d\mu < \infty$. For each $x \in X$ denote

$$L(x) = \sup\{n;\ f_n(x) > 1\}$$

Prove that $L(x) < \infty$, μ-a.e.

3.) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Prove that for every $\varepsilon > 0$, there is a bounded interval I such that $\left|\int_E f\,dx\right| < \varepsilon$ for every measurable set E with $E \cap I = \phi$.

4.) Let F be a Banach space and $\mathcal{F} \subset \Sigma$ a sub σ-algebra. Prove the existence of the conditional expectation $E(f|\mathcal{F})$ for every μ-integrable function $f: X \to F$.

5.) Let $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ be two real measure spaces. Prove the existence of the product measure $\mu \times \nu$ and that $|\mu \times \nu| = |\mu| \times |\nu|$.

6.) Let μ be the Lebesgue measure on $\mathbb{R}$ and f a Lebesgue integrable function on $\mathbb{R}$ such that $\int_0^x f\, d\mu = 0$ for every $x \in \mathbb{R}$ (if $x < 0$, $\int_0^x = -\int_x^0$). What can you say about f?

7.) Let μ be the Lebesgue measure on $\mathbb{R}$ and $f_n = -\varphi_{(n,\infty)}$, $n = 1, 2, \ldots$. Show that (f_n) is increasing but that the conclusion of the monotone convergence theorem is not true. Explain why.

8.) Let μ be the Lebesgue measure on $(0,1)$. Each point $x \in (0,1)$ has a binary expansion

$$x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}, \quad \text{where } x_i = 0 \text{ or } 1.$$

(The expansion is unique except a countable set, which we can throw out of the space).

(i). Define $T_k(x) = x_k$. Sketch the graphs of T_1, T_2, T_3.

(ii) Let $\mathcal{F}_k = \sigma(T_1, T_2, \ldots, T_k)$. Describe $\mathcal{F}_k$.

(iii) Let $f \in L^1(\mu)$. Find a function g_1 which is $\mathcal{F}_1$-measurable and satisfies

$$\int_A g_1\, d\mu = \int_A f\, d\mu \quad \text{for } A \in \mathcal{F}_1$$

(i.e. find $g_1 = E(f|\mathcal{F}_1)$). Find $g_2 = E(f|\mathcal{F}_2)$.