Ph.D. Exam in Measure Theory and Analysis
May, 1991

Be sure to give complete proofs and justify your steps. Do not leave gaps.

1. State:
   (a) The Lebesgue Dominated Convergence Theorem
   (b) Fatou's Lemma
   (c) The Radon-Nikodym Theorem
   (d) $(L^1(S,\Sigma,\mu)^* = L^\infty(S,\Sigma,\mu)$

2. State and prove the Vitali Convergence Theorem for integrals (convergence in measure form).

3. Show that a subset A of the reals is a set of Lebesgue measure zero if and only if there exists a sequence of intervals I_m such that

   (a) $\sum\limits_m$ length $(I_m) < \infty$.

   (b) $A \subset \bigcup\limits_{m=k}^{\infty} I_m$ for every k.

4. Let f be Lebesgue integrable on $\mathbb{R}$. Suppose that $\int\limits_0^x f dm = 0$ for every $x \in \mathbb{R}$. What can you conclude about f?

5. State and prove Fubini's Theorem.

6. Let $(S_m, \Sigma_m, P_m)_{m=1}^{\infty}$ be a sequence of probability spaces. Let $E_m \in \Sigma_m$, $m = 1, 2, \ldots$.

   (a) Show that $E = \mathop{X}\limits_{m=1}^{\infty} E_m \in \bigotimes\limits_{n=1}^{\infty} \Sigma_m$.

   (b) Prove that $\left(\bigotimes\limits_{m=1}^{\infty} P_m\right)(E) = \prod\limits_{m=1}^{\infty} P_m(E_m)$