Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and circle the numbers of the seven problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

Note. Below *ring* means associative ring with identity, and *module* means unital module unless otherwise specified.

1. (10 points) Let K be a finite field, and let F be a subfield of K. Prove that the extension K/F is Galois.

2. Let $p(x) = x^7 - 3 \in \mathbf{Q}[x]$, $\zeta \in \mathbf{C}$ a primitive 7-th root of 1, $\xi = \sqrt[5]{3} \in \mathbf{C}$, and K the splitting field of $p(x)$.
   (a) (4 points) Compute the Galois group of $K/\mathbf{Q}$.
   (b) (3 points) Compute the Galois group of $K(\zeta)/\mathbf{Q}(\zeta)$.
   (c) (3 points) Compute the Galois group of $K(\xi)/\mathbf{Q}(\xi)$.

   In every case, carefully justify that your calculation is correct.

3. (10 points) Let A and B be abelian groups, and $T = A \otimes_{\mathbf{Z}} B$. Let $\phi\colon A \to A$ be a group homomorphism. Prove that there is a group homomorphism

$$\psi\colon T \to T$$

   such that, for all $a \in A$ and $b \in B$, we have

$$\psi\,(a \otimes_{\mathbf{Z}} b) = \phi(a) \otimes_{\mathbf{Z}} b.$$

4. Let $\mathcal{F}G$ be the category of all finite groups.
   (a) (5 points) Recall the definition of *free object* in an arbitrary concrete category $\mathcal{C}$.
   (b) (5 points) Prove from your definition that there exists a finite set S such that there is no free object in $\mathcal{F}G$ freely generated by S.

5. (10 points) Give an example of a projective module which is not free. Prove that your example has the desired properties.

6. (10 points) State and prove Hilbert's Basis Theorem.

7. (10 points) Prove the *Lying-Over Theorem*: Let S be an integral extension of an integral domain R, and let P be a prime ideal of R. Then, there exists a prime ideal Q of S, such that $Q \cap R = P$.

8. (10 points) Let R be an integral domain that is a local ring. Let I be an invertible ideal of R. Prove that I is principal.

9. (10 points) Let R be the ring of all rational numbers with odd denominators. Prove that the Jacobson radical $J(R)$ of R consists of all fractions whose denominator is odd and whose numerator is even.

10. (a) (5 points) Define what is a *Dedekind domain*.

    (b) (2 points) Give an example of a Dedekind domain which is not a unique factorization domain.

    (c) (3 points) Prove that your example is a Dedekind domain and not a unique factorization domain carefully explaining any standard results you use.

11. (10 points) Let R be a ring with identity. Consider the ring $\mathrm{Hom}_R(R, R)$ of left R-module homomorphisms from R to itself. Prove that $\mathrm{Hom}_R(R, R)$ is isomorphic, as a ring, to R^{op}.