Ph.D. Exam: Numerical Analysis, May, 2020.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. **(a)** Let $A \in \mathbb{C}^{m\times n}$. Prove or give a counterexample: Every matrix norm that is induced by a vector norm satisfies the submultiplicative property $\|AB\| \leq \|A\|\|B\|$. If you prove this, make sure to justify each nontrivial step.

(b) Let $A \in \mathbb{C}^{m\times m}$. Prove that $\|A\|_2 = (\rho(A^*A))^{1/2}$, where $\rho(A)$ is the spectral radius of A.

2. Suppose A is Hermitian positive definite.

(a) Prove that each principal submatrix of A is Hermitian positive definite.

(b) Prove that an element of A with largest magnitude lies on the diagonal.

(c) Prove that A has a Cholesky decomposition.

3. Let $A \in \mathbb{C}^{m\times m}$ be Hermetian.

(a) Show that all eigenvalues of A are real.

(b) Define the stationary iterative method (a.k.a. fixed point method)

$$x^{(k+1)} = Ax^{(k)} + b. \tag{1}$$

Suppose (1) has fixed-point x, namely x satisfies $x = Ax+b$. Show the iteration (1) converges to x from any starting guess $x^{(0)}$, that is $x^{(k)} \to x$ as $k \to \infty$, if and only if the eigenvalues λ_i of A satisfy $|\lambda_i| < 1,\ i = 1, \dots, m$. You may use the fact that Hermetian matrix A is unitarily diagonalizable.

4. Suppose the 5×5 symmetric matrix A has eigenvalues known to within the given tolerances.

$$\begin{aligned} 3.5 &> \lambda_1 > 2.5 \\ 2.0 &> \lambda_2 > 1.0 \\ 1.0 &> \lambda_3 > -1.0 \\ -1.0 &> \lambda_4 > -2.0 \\ -2.5 &> \lambda_5 > -3.5. \end{aligned}$$

(a) Describe how shifting can be used so that the power method can be used to compute λ_1 with guaranteed convergence. Clearly explain your choice of shift.

(b) Provide an upper bound for the convergence rate using the shift you chose in **(a)** for λ_1. Is there another shift that would decrease this worst-case convergence rate?

5. For $x, y > 0$, consider computing $f(x,y) = \sqrt{y + x^2} - \sqrt{y}$ in floating-point arithmetic with machine precision ϵ_m.

(a) Explain the difficulties in computing $f(x,y)$, if $x \ll y$. What are the absolute and relative errors if $x^2/y < \epsilon_m$, if $f(x,y)$ is computed directly from the form given above?

(b) Suppose $x^2/y < \epsilon_m$. Describe a way to compute $f(x,y)$ with more accuracy in this situation.

6. Let $\alpha > 0$. For (i) $p = 2$; (ii) $p = \infty$, find the constant c_p that minimizes

$$E_p(c) = \|t^\alpha - c\|_p = \left(\int_0^1 |t^\alpha - c|^p \, \mathrm{d}\, t\right)^{1/p},$$

and find $E_p(c_p)$, for each of those values of p.

7. Let $\{\phi_k\}_{k=0}^{m+1}$ be the set of monic orthogonal polynomials with respect to inner product $(u, v) = \int_a^b u(x)v(x)w(x) \, \mathrm{d}\, x$. Let $\phi_{-1} = 0$ and $\phi_0 = 1$.

(a) Find expressions for constants α_k and β_k to determine a 3-term recurrence relation of the form

$$\phi_{k+1}(x) = (x - \alpha_k)\phi_k - \beta_k \phi_{k-1}(x), \; k = 0, \dots, m.$$

(b) Use the above relation to determine ϕ_1, ϕ_2 and ϕ_3, using the inner-product $(u, v) = \int_0^1 u(x)v(x) \, \mathrm{d}\, x$.

8. Consider finding a zero of function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ that can be written as the sum of a linear and nonlinear part

$$f(x) = Bx + G(x),$$

where B is a nonsingular matrix and $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinear function. At a point x_k consider an affine model $M_k(x) = a_k + A_k(x - x_k)$, where the quantities $a_k \in \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n\times n}$ are to be determined.

(a) Determine a_k and A_k so that the following conditions hold:

$$M_k(x_k) = f(x_k) \;\text{ and }\; M_k'(x_k) = B.$$

(b) Derive the iteration obtained by defining x_{k+1} as the zero of $M_k(x)$.

9. Let $f \in C^\infty(a, b)$. Let $x_0 < x_1 < x_2$ be three points in $[a, b]$ that are not necessarily equally spaced.

(a) Based the quadratic interpolant p_2 which satisfies $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$, find the centered finite difference approximations to $f'(x_1)$ and $f''(x_1)$ (you should explicitly show how the difference approximations are derived from the interpolant).

(b) Derive an expression for the error, $f'(x_1) - p_2'(x_1)$.

10. **(a)** For $f \in C^\infty(a, b)$, the composite trapezoidal quadrature rule with n subintervals with length $h = (b - a)/n$ satisfies

$$\left|\int_a^b f(x) \, \mathrm{d}\, x - I_{T,n}\right| = a_2h^2 + a_4h^4 + a_6h^6 + \dots,$$

where the coefficients $a_2, a_4, \dots$, do not depend on n.

Find and inductively prove an expression for the kth Richardson extrapolant I_k of $I_0 := I_{T,n}$.

(b) Consider a quadrature formula of the type

$$\int_0^1 f(x) \, \mathrm{d}\, x \approx \alpha f(x_1) + \beta[f(1) - f(0)].$$

Determine α, β and x_1 such that the degree of exactness is as large as possible. What is the maximum degree of exactness?