**Variational Analysis/Numerical Optimization PhD Qualifier – May 6, 2025**

**Do 8 problems, 4 from problems 1–5 and 4 from problems 6–10.**

**Call 352-256-9514 if any questions arise.**

1. Recall that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if

$$f[(1-\alpha)\mathbf{x} + \alpha\mathbf{y}] \le (1-\alpha)f(\mathbf{x}) + \alpha f(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \text{ and } \alpha \in [0,1].$$

   (a) If $f$ is continuously differentiable, then show that $f$ is convex if and only if

$$f(\mathbf{y}) \ge f(\mathbf{x}) + \nabla f(\mathbf{x})(\mathbf{y} - \mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

   (b) If $f$ is twice continuously differentiable, then show that $f$ is convex if and only if $\nabla^2 f$ is positive semidefinite on $\mathbb{R}^n$.

2. Suppose that $\mathbf{\Phi} : \mathcal{K} \to \mathbb{R}^n$ is continously differentiable on $\mathcal{K} \subset \mathbb{R}^n$.

   (a) If $\mathcal{K}$ is a convex set, then show that

$$\|\mathbf{\Phi}(\mathbf{x}) - \mathbf{\Phi}(\mathbf{y})\| \le \mu\|\mathbf{x} - \mathbf{y}\|$$

   for all $\mathbf{x}$ and $\mathbf{y} \in \mathcal{K}$ where $\mu$ is the supremum of the singular values of $\nabla\mathbf{\Phi}$ over $\mathcal{K}$.

   (b) If $\mathbf{\Phi}$ is a contraction on $\mathcal{K}$, where $\mathcal{K}$ is a closed set, and $\mathbf{\Phi}(\mathbf{x}) \in \mathcal{K}$ for each $\mathbf{x} \in \mathcal{K}$, then then show that $\mathbf{\Phi}$ has a unique fixed point in $\mathcal{K}$.

3. Let $\mathbf{U}$ and $\mathbf{V} \in \mathbb{R}^{n\times m}$ and $\mathbf{M} \in \mathbb{R}^{n\times n}$ with $\mathbf{M}$ invertible.

   (a) If $\mathbf{I} + \mathbf{V}^\mathsf{T}\mathbf{M}^{-1}\mathbf{U}$ is invertible, then show that $\mathbf{M} + \mathbf{U}\mathbf{V}^\mathsf{T}$ is invertible with

$$(\mathbf{M} + \mathbf{U}\mathbf{V}^\mathsf{T})^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{V}^\mathsf{T}\mathbf{M}^{-1}\mathbf{U})^{-1}\mathbf{V}^\mathsf{T}\mathbf{M}^{-1}.$$

   (b) If $\mathbf{U} = \mathbf{V} = \mathbf{A} \in \mathbb{R}^{n\times m}$ where the columns of $\mathbf{A}$ are linearly independent, and $\mathbf{M} = \mathbf{Q}$, a symmetric matrix, then show that

$$(\mathbf{Q}+p\mathbf{A}\mathbf{A}^\mathsf{T})(\mathbf{I}+p\mathbf{A}\mathbf{A}^\mathsf{T})^{-1} = \mathbf{Q}_0+\mathcal{O}(1/p), \quad \text{where } \mathbf{Q}_0 = \mathbf{Q}+(\mathbf{I}-\mathbf{Q})[\mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}].$$

4. Consider the following quadratic program:

$$\min \; q(\mathbf{x}) := \sum_{i=1}^{n} 0.5x_i^2 + c_i x_i \quad \text{subject to } \mathbf{a}^\mathsf{T}\mathbf{x} = b, \; \mathbf{x} \ge \mathbf{0}, \qquad \text{(QP)}$$

   where $\mathbf{c}$ and $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a} \ge \mathbf{0}$, and $b > 0$ is a scalar.

(a) Develop an algorithm for solving (QP) by maximizing the dual function

$$L(\lambda) = \inf \ \{q(\mathbf{x}) + \lambda(\mathbf{a}^{\mathsf{T}}\mathbf{x} - b) : \mathbf{x} \geq \mathbf{0}\}$$

over the dual multiplier $\lambda$. Here, the constraint $\mathcal{K}$ in the dual function is taken as $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}$.

(b) Given the optimal $\lambda$ for the dual problem, what is the solution of the primal problem?

5. Consider the primal problem

$$\min \ f(\mathbf{x}) \quad \text{subject to } \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}, \ \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and the components of $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^l$ are convex and continuously differentiable, $\mathbf{A} \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$. The dual problem is

$$\max \ L(\boldsymbol{\lambda}, \boldsymbol{\mu}) \quad \text{subject to } \boldsymbol{\lambda} \in \mathbb{R}^m, \ \boldsymbol{\mu} \in \mathbb{R}^l, \ \boldsymbol{\mu} \geq \mathbf{0}, \tag{D}$$

where

$$\begin{aligned} L(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \inf \ \{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) : \mathbf{x} \in \mathbb{R}^n\}, \\ \mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= f(\mathbf{x}) + \boldsymbol{\lambda}^{\mathsf{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) + \boldsymbol{\mu}^{\mathsf{T}}\mathbf{g}(\mathbf{x}). \end{aligned}$$

(a) Show that if (P) has a local minimizer $\mathbf{x}^*$, then $\mathbf{x}^*$ is a global minimizer for (P). Moreover, if the first-order optimality conditions hold at $\mathbf{x}^*$, then there is a solution to (D) with no duality gap.

(b) Returning to (QP) of problem 4, show that there is no duality gap. You may wish to consider both the set

$$\mathcal{S} = \{(r, s) \in \mathbb{R}^2 : r \geq q(\mathbf{x}) \text{ and } s = \mathbf{a}^{\mathsf{T}}\mathbf{x} - b \text{ for some } \mathbf{x} \geq \mathbf{0}\},$$

and the point $(q^*, 0)$ where $q^*$ is the optimal cost for (QP). Explain why this point does not lie in the interior of $\mathcal{S}$. Use the separating hyperplane theorem to separate $\mathcal{S}$ and the point, and then analyze the resulting inequality.

6. Consider the initial-value problem

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{a}$$

where $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{f} : \mathcal{B}_\delta(\mathbf{a}) \times \mathcal{U} \to \mathbb{R}^n$ with $\mathcal{U} \subset \mathbb{R}^m$ a compact set and $\delta > 0$. It is assumed that $\mathbf{f}$ is continuous on $\mathcal{B}_\delta(\mathbf{a}) \times \mathcal{U}$ and uniformly Lipschitz continuous in its first argument with Lipschitz constant $L$ satisfying

$$|\mathbf{f}(\mathbf{x}_1, \ \mathbf{u}) - \mathbf{f}(\mathbf{x}_2, \ \mathbf{u})| \leq L|\mathbf{x}_1 - \mathbf{x}_2|$$

for all $\mathbf{x}_1, \ \mathbf{x}_2 \in \mathcal{B}_\delta(\mathbf{a})$ and $\mathbf{u} \in \mathcal{U}$. If $\epsilon = \delta/M$ where

$$M = \max \ \{|\mathbf{f}(\mathbf{x}, \ \mathbf{u})| : \mathbf{x} \in \mathcal{B}_\delta(\mathbf{a}), \ \mathbf{u} \in \mathcal{U}\},$$

then show that the initial-value problem has a unique, absolutely continuous solution on the interval $[0, \epsilon]$ with $\mathbf{x}(t) \in \mathcal{B}_\delta(\mathbf{a})$ for all $t \in [0, \epsilon]$.

7. Consider the equation

$$(P(x)u'(x))' = Q(x)u(x), \quad x \in [0,1], \tag{E}$$

where $Q$ is continuous and $P$ is continuously differentiable and strictly positive on $[0,1]$. Show that if the solution to the initial-value problem for (E) with $u(0) = 0$ and $u'(0) = 1$ is strictly positive on the half-open interval $(0,1]$, then there exists a strictly positive solution to the second-order differential equation (E) (without boundary conditions at $x = 0$ or $x = 1$).

8. Consider the variational problem

$$\min \int_0^1 \frac{1}{2}u'(x)^2 + e^{u(x)} dx \quad \text{subject to } u \in \mathcal{H}_0^1.$$

   (a) What is the first-order necessary optimality condition (Euler equation) for this variational problem?

   (b) Consider a uniform mesh $x_k = kh$ where $h = 1/N$; let $u_k$ denote an approximation to $u(x_k)$. Of course, $u_0 = u_N = 0$. Give a finite difference approximation in terms $u_1, \ldots, u_{N-1}$ to the solution of the Euler equation.

   (c) Let $\mathbf{F}(\mathbf{u})$ denote the finite difference system where $\mathbf{u} = (u_1, u_2, \ldots, u_{N-1})^{\mathsf{T}} \in \mathbb{R}^{N-1}$, and let $\mathbf{u}^*$ denote the vector formed by evaluating the solution of the Euler equation at the mesh points $x_1, x_2, \ldots, x_{N-1}$. Obtain a bound for the components of $\mathbf{F}(\mathbf{u}^*)$ in terms of the mesh spacing $h$.

9. Consider the initial-value problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{0},$$

where $\mathbf{x} : [0,1] \to \mathbb{R}^n$ and $\mathbf{A} \in \mathcal{L}^\infty([0,1]; \mathbb{R}^{n \times n})$.

   (a) Assuming $\mathbf{u} \in \mathcal{L}^2$, obtain a bound for the sup-norm of $\mathbf{x}$ in terms of the $L^2$ norm of $\mathbf{u}$.

   (b) Let $\Omega : \mathcal{H}_0^1 \to \mathbb{R}$ be defined by

$$\Omega(h) = \frac{1}{2} \int_0^1 r^2(x)\, dx \quad \text{where } r(x) = h'(x) + w(x)h(x).$$

   Show that there exists $\beta > 0$ such that

$$\Omega(h) \geq \beta(\|h\|^2 + \|h'\|^2) \quad \text{for all } h \in \mathcal{H}_0^1,$$

   where $\|\cdot\|$ is the $\mathcal{L}^2$ norm.

(c) Consider the quadratic programming problem

$$\min \frac{1}{2}\mathbf{x}^\mathsf{T}\mathbf{R}\mathbf{x} + \mathbf{b}^\mathsf{T}\mathbf{x} \quad \text{subject to } \mathbf{x} \in \mathcal{K},$$

where $\mathcal{K} \subset \mathbb{R}^n$ is a closed convex set and $\mathbf{R} \in \mathbb{R}^{n\times n}$ is positive definite with smallest eigenvalue $\alpha > 0$. Show that the solutions $\mathbf{u}_i$, $i = 1, 2$, associated with $\mathbf{b} = \mathbf{b}_i$, $i = 1, 2$ respectively, satisfy

$$\|\mathbf{u}_1 - \mathbf{u}_2\| \leq \|\mathbf{b}_1 - \mathbf{b}_2\|/\alpha.$$

10. Consider the following control problem:

$$\min \int_0^1 \frac{1}{2}u^2(x) + y(x)\, dx \;\; \text{subject to } y'(x) = u(x),\;\; y(0) = 0,\;\; u(x) \geq \ell(x),$$

where $\ell(x)$ is a given lower bound for the control $u(x)$ at each $x \in [0, 1]$.

(a) What is the first-order optimality condition (Pontryagin minimum principle) for this problem?

(b) What is the solution of the control problem?