Variational Analysis/Numerical Optimization PhD Qualifier – May 6 or 7, 2026

Do 8 problems, 4 from problems 1–5 and 4 from problems 6–10.

Call 352-256-9514 if any questions arise.

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic:

$$f(x) = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} - \mathbf{b}^{\mathsf{T}}\mathbf{x}.$$

It is assumed that $\mathbf{Q}$ is symmetric and positive definite, and $\mathbf{x}^*$ denotes the unique minimizer of f. Consider the steepest descent algorithm with a Cauchy line search (in each iteration, $\mathbf{x}_{k+1}$ is obtained from $\mathbf{x}_k$ using an exact minimization in the search direction). Show that if at any iterate $\mathbf{x}_k$, the error $\mathbf{x}^* - \mathbf{x}_k$ is an eigenvector of $\mathbf{Q}$, then $\mathbf{x}_{k+1} = \mathbf{x}^*$.

2. Use duality theory to prove the arithmetic-geometric mean inequality:

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$

for all $\mathbf{x} \geq \mathbf{0}$.

3. Suppose that $\mathbf{x}^*$ is a local minimizer of $f : \mathbb{R}^n \to \mathbb{R}$. If f is a smooth function with $\nabla^2 f(\mathbf{x}^*)$ positive definite, then show that for all starting guesses $\mathbf{x}_0$ sufficiently close to $\mathbf{x}^*$, $\mathbf{x}_k$ converges to $\mathbf{x}^*$ at a quadratic rate.

4. Suppose $\mathbf{A} \in \mathbb{R}^{m\times n}$ with the rows of $\mathbf{A}$ linearly independent and $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is symmetric. Show that

$$(\mathbf{Q} + p\mathbf{A}^{\mathsf{T}}\mathbf{A})(\mathbf{I} + p\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1} = \mathbf{B} + \mathcal{O}(p^{-1}),$$

as p tends to infinity, where $\mathbf{B} \in \mathbb{R}^{n\times n}$ is independent of p. Recall the Woodbury formula: If $\mathbf{M}$ and $(\mathbf{I} + \mathbf{V}\mathbf{M}^{-1}\mathbf{U}^{\mathsf{T}})$ are invertible for some $\mathbf{U}$ and $\mathbf{V} \in \mathbb{R}^{m\times n}$, then

$$(\mathbf{M} + \mathbf{U}^{\mathsf{T}}\mathbf{V})^{-1} = \mathbf{M}^{-1} - \mathbf{M}^{-1}\mathbf{U}^{\mathsf{T}}(\mathbf{I} + \mathbf{V}\mathbf{M}^{-1}\mathbf{U}^{\mathsf{T}})^{-1}\mathbf{V}\mathbf{M}^{-1}.$$

5. Consider the quadratic program

$$\min \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} - \mathbf{b}^{\mathsf{T}}\mathbf{x} \text{ subject to } \mathbf{a}^{\mathsf{T}}\mathbf{x} = c, \tag{1}$$

where $\mathbf{A}$ is a symmetric, positive definite matrix. The penalty approximation to (1) is

$$\min \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} - \mathbf{b}^{\mathsf{T}}\mathbf{x} + \lambda(\mathbf{a}^{\mathsf{T}}\mathbf{x} - c) + \frac{p}{2}(\mathbf{a}^{\mathsf{T}}\mathbf{x} - c)^2, \tag{2}$$

where $p > 0$ is the penalty parameter and λ is any approximation to the Lagrange multiplier of (1) associated with the constraint $\mathbf{a}^{\mathsf{T}}\mathbf{x} - c = 0$. Use the first-order optimality conditions for both (1) and (2) to obtain a bound for the distance between the solution $\mathbf{x}^*$ and multiplier λ^* of (1) and the solution $\mathbf{x}$ and multiplier approximation $\lambda + p(\mathbf{a}^{\mathsf{T}}\mathbf{x} - c)$ gotten from (2).

6. Find a stationary point for the variational problem:

$$\min \int_0^1 \frac{1}{2}y'(x)^2 + y'(x)y(x) + y'(x) + y(x)\ dx \quad \text{subject to } y \text{ continuously differentiable.}$$

   Note that there are no constraints on $y(x)$ at $x = 0$ or at $x = 1$. Hence, both boundary terms need to be taken into account in the analysis of the first-order optimality conditions; these boundary terms lead to two "natural" boundary conditions in the first-order optimality conditions.

7. Suppose that P and Q are real-valued functions defined on the interval $[0, 1]$ with P continuously differentiable and Q continuous. It is assumed that there exists $\alpha > 0$ such that $P(x) \geq \alpha$ for all $x \in [0, 1]$. Define Γ by

$$\Gamma(h) := \int_0^1 P(x)h'(x)^2 + Q(x)h(x)^2\ dx.$$

   Suppose that Γ has the following property: There exists $\gamma > 0$ such that

$$\Gamma(h) \geq \gamma \int_0^1 h'(x)^2 + h(x)^2\ dx \quad \text{for all } h \in \mathcal{H}_0^1,$$

   where $\mathcal{H}_0^1$ is the space of functions with h' square integrable over $[0, 1]$ and $h(0) = h(1) = 0$. Prove the following: If u is a solution of the boundary-value problem

$$\frac{d}{dx}(P(x)u'(x)) = Q(x)u(x) \quad \text{for all } x \in [0, a] \text{ with } u(0) = 0 = u(a),$$

   where $a \in (0, 1)$ is given, then u vanishes on the entire interval $[0, a]$. Hint: extend u from $[0, a]$ to $[0, 1]$ by setting $u(x) = 0$ for $x \in [a, 1]$. Then integrate by parts to show that $\Gamma(u) = 0$.

8. Consider the initial-value problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{0},$$

   where $\mathbf{x} : [0, 1] \to \mathbb{R}^n$ and $\mathbf{A} \in \mathcal{L}^\infty([0, 1]; \mathbb{R}^{n\times n})$. Obtain a bound for the sup-norm of $\mathbf{x}$ in terms of the $\mathcal{L}^2$ norm of $\mathbf{u}$. Hint: Recall Grönwall's inequality

$$x(t) \leq \alpha(t) + L\int_0^t x(s)ds \quad \text{on } [0,\ a] \quad \Rightarrow \quad x(t) \leq \alpha(t)e^{Lt} \text{ on } [0,\ a]$$

   where α is monotone nondecreasing.

9. Suppose $\Phi : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K}$ is a closed subset of a Banach space, and Φ is a contraction on $\mathcal{K}$. Show that the iteration $x_{k+1} = \Phi(x_k)$, $k \geq 0$, where $x_0 \in \mathcal{K}$, converges to the unique fixed point x^* of Φ in $\mathcal{K}$. Obtain a bound for $\|x_k - x^*\|$ in terms of $\|x_0 - x^*\|$.

10. Consider the variational problem

$$\min \int_0^1 \frac{1}{2}u'(x)^2 + e^{u(x)}dx \quad \text{subject to } u \in \mathcal{H}_0^1. \tag{3}$$

    (a) What is the first-order necessary optimality condition (Euler equation) for a local minimizer of this variational problem?

    (b) Show that if a minimizer of (3) exists, then it is unique.