Topology Ph.D Exam August 20th 2024

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper (10 pts each).

1. Let $X = \mathbb{R}/\mathbb{Z}$ be the quotient space obtained from the real line $\mathbb{R}$ by collapsing the set of integers to a point.

(a) Is X metrizable?

(b) Does X admit a CW complex structure?

(c) Compute the fundamental group of X.

(d) Compute homology groups of X.

2. (a) Prove that the 3-dimensional real projective space $\mathbb{R}P^3$ is orientable.

(b) Does there exist a degree 2 continuous map $f : \mathbb{R}P^3 \to T^3$ where $T^3 = S1 \times S^1 \times S^1$ is a 3-dimensional torus?

(c) Does there exist a degree 2 continuous map $f : T^3 \to \mathbb{R}P^3$?

3. Let $A \subset \mathbb{R}^2$ be an infinite countable subspace.

(a) Can A be connected?

(b) Is $\mathbb{R}^2 \setminus A$ connected?

4. Does there exist a closed orientable n-manifold M with

(a) $n = 4$ and $H_2(M) = \mathbb{Z}$?

(b) $n = 5$ and $H_3(M) = \mathbb{Z}$?

(c) $n = 6$ and $H_3(M) = \mathbb{Z}$?

5. Prove that the fundamental group of a topological group is abelian.

Answer the following with complete definitions or statements or short proofs (5 pts each)

6. Compute the Euler characteristic of the manifold $M = \mathbb{H}P^3 \times \mathbb{C}P^4 \times \mathbb{R}P^5$.

7. State the Lefschetz Fixed Point Theorem. Derive the Brouwer Fixed Point Theorem from the Lefschetz Fixed Point Theorem.

8. State the Baire Category Theorem.

9. Suppose that A is a retract of X. Show that that $H_i(A)$ is a direct summand of $H_i(X)$.

10. Give the definition of the mapping cylinder. Is the Mobius band a mapping cylinder?

11. Is the 2-sphere S^2 with k points removed homeomorphic to a topological group for

(a) $k = 1$? (b) $k = 2$? (c) $k = 3$?

12. Give the definition of the mapping cone. Is there a map $f : X \to Y$ with the mapping cone C_f homeomorphic to $\mathbb{R}P^3$?

13. Which of the following spaces are topological groups?

(a) the torus T^2;

(b) the Klein bottle K;

(c) the projective space $\mathbb{R}P^3$.

14. Is $\mathbb{R}^\omega$ connected in the uniform topology?

15. State the Five Lemma.