Topology Ph.D Exam

May 2024

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper

1. Prove that compact Hausdorff space is normal.

2 .Let X_α be a family of connected spaces that have a common point. Prove that $\bigcup X_\alpha$ is connected.

3. Does there exist a closed orientable n-manifold M with

(a) $n = 4$ and $H_2(M) = \mathbb{Z}$?

(b) $n = 5$ and $H_3(M) = \mathbb{Z}$?

(c) $n = 6$ and $H_3(M) = \mathbb{Z}$?

4. Show that the Mobius strip is not a retract of its boundary.

5. Is the space of all real $n \times n$-matrices compact? Connected?

Answer the following with complete definitions or statements or short proofs.

6. Explain relations between connectedness and path connectedness.

7. State the Lefschetz Fixed Point Theorem. Explain why every continuous map $D^n \to D^n$ has a fixed point.

8. Give definitions of retraction, deformation retraction, retract, and deformation retract. Give an example of a retract A of a topological space which is not a deformation retract.

9. Show that the spaces $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ are not homotopy equivalent.

10. Give definitions of the mapping cylinder and the mapping cone. Is the Mobius strip a mapping cylinder?

11. Formulate the Urysohn Lemma.

12. Does $S^1 \vee S^1$ admit a covering space with an abelian fundamental group?

13. What is the Euler characteristic of the product of five 2-dimensional spheres $S^2 \times S^2 \times S^2 \times S^2 \times S^2$?

14. Is $\mathbb{R}^\omega$ connected in the uniform topology?

15. State the Five Lemma.