**Ph.D. Examination – Topology**
**August 2022**

**Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.**

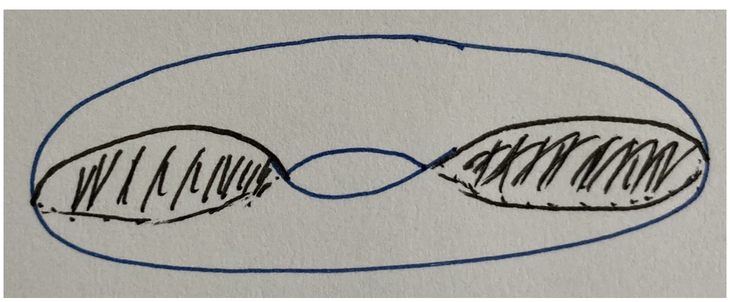
1. Suppose $M$ is a compact 5-manifold with $H_0(M;\mathbb{Z}) = \mathbb{Z}$, $H_1(M;\mathbb{Z}) = \mathbb{Z}_3$, and $H_2(M;\mathbb{Z}) = \mathbb{Z}$. Is $M$ orientable? What are $H_3(M;\mathbb{Z})$, $H_4(M);\mathbb{Z}$, and $H_5(M;\mathbb{Z})$?

2. Let $M$ be a compact, connected, nonorientable 3-manifold. Prove that $H_1(M;\mathbb{Z})$ is infinite.

3. Let $X$ be a connected metric space with metric $d$ and let $p \in X$. Prove that if $X \setminus \{p\} \neq \emptyset$, then $X \setminus \{p\}$ is not compact.

4. Let $\Sigma_g$ denote the closed surface of genus $g$, and let $\Sigma_{g,r}$ denote the surface of genus $g$ with $r$ boundary components (i.e., $\Sigma_g$ with $r$ discs removed). Which of the surfaces $\Sigma_{g,r}$ can admit a map $f : \Sigma_{g,r} \to \Sigma_{g,r}$ homotopic to the identity that does not have a fixed point?

5. Let $X$ be the space obtained by filling in two discs in the torus, as shown (the discs lie inside the surface of the torus). Compute $H_\bullet(X;\mathbb{Z})$.



**Answer the following with complete definitions, statements, or short proofs.**

6. Prove or give a counterexample: If $A \subset X$ is path-connected, then the closure $\overline{A}$ is path-connected.

7. Compute $\chi(\mathbb{R}P^2 \times \mathbb{C}P^3 \times S^4)$

8. Give the definition of a normal topological space. Show that a smooth manifold is a normal space.

9. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

10. Prove that if $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

11. Prove that the torus $T^2 = S^1 \times S^1$ is not homotopy equivalent to $S^1 \vee S^1 \vee S^2$.

12. State the Tietze Extension Theorem.

13. Describe all the connected covering spaces $E \to \mathbb{R}P^2 \vee \mathbb{R}P^3$.

14. Does the following exact sequence of abelian groups necessarily split? Prove or give a counterexample.

$$0 \to \mathbb{Z}_2 \to A \to \mathbb{Z}_2 \to 0$$

15. Compute the integral homology of the space $\mathbb{C}P^2 \times \mathbb{R}P^2$.