Topology Ph.D Exam May 2021

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper (10 pts each).

1. Is the quotient space $X = \mathbb{R}/\mathbb{Z}$ obtained from the reals $\mathbb{R}$ by collapsing the integers to a point metrizable?

Compute homology groups of X.

2. Show that any continuous map $f : \mathbb{R}P^3 \to T^3$ is null-homotopic.

3. Let $A \subset \mathbb{R}^3$ be the union of a countable family of lines in $\mathbb{R}^3$. Prove that $\mathbb{R}^3 \setminus A$ is connected.

4. Does there exist a closed orientable n-manifold M with

(a) $n = 4$ and $H_2(M) = \mathbb{Z}$?

(b) $n = 5$ and $H_3(M) = \mathbb{Z}$?

(c) $n = 6$ and $H_3(M) = \mathbb{Z}$?

5. Let $p : X \to M_2$ be a covering map of an orientable surface of genus 2 that corresponds to the subgroup $G \subset \pi_1(M_2)$ generated by $\{a_1, b_1\}$ in

$$\pi_1(M_2) = \langle a_1, a_2, b_1, b_2 \mid [a_1, b_1][a_2 b_2] = 1\rangle.$$

Compute homology groups of X.

Answer the following with complete definitions or statements or short proofs (5 pts each)

6. Compute the Euler characteristic of the manifold $M = S^2 \times S^4 \times \mathbb{C}P^{51} \times \mathbb{R}P^{99}$.

7. State the Lefschetz Fixed Point Theorem. Derive the Brouwer Fixed Point Theorem from the Lefschetz Fixed Point Theorem

8. Give definitions of retraction, deformation retraction, retract, and deformation retract. Give an example of a retract A of a topological space which is not a deformation retract.

9. Suppose that A is a retract of X. Is it true that $H_i(A)$ is a direct summand of $H_i(X)$? What about $H^i(A)$ and $H^i(X)$?

10. Give definition of the mapping cylinder. Is the Mobius band a mapping cylinder?

11. Formulate the Urysohn Lemma.

12. Give definition of the mapping cone. Is there a map $f : X \to Y$ with the mapping cone C_f homeomorphic to $\mathbb{R}P^3$?

13. Which of the following spaces are topological groups?

(a) the torus T^2;

(b) the Klein bottle K;

(c) the projective space $\mathbb{R}P^3$.

14. Is $\mathbb{R}^\omega$ connected in the uniform topology?

15. State the Five Lemma.