Ph.D. Examination – Topology
May 2020

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Let X be a nonempty set. A nonempty collection Φ of subsets $U \subseteq X \times X$ is a *uniformity* if it satisfies the following axioms:
   (1) If $U \in \Phi$, then $\Delta \subseteq U$, where Δ is the diagonal $\{(x, x) : x \in X\}$.
   (2) If $U \in \Phi$ and $U \subseteq V \subseteq X \times X$, then $V \in \Phi$.
   (3) If $U, V \in \Phi$, then $U \cap V \in \Phi$.
   (4) If $U \in \Phi$, then there is a $V \in \Phi$ with $V \circ V \subseteq U$, where $A \circ B = \{(x, z) | \exists y \in X$ such that $(x, y) \in A$ and $(y, z) \in B\}$.
   (5) If $U \in \Phi$, then $U^{-1} \in \Phi$, where $U^{-1} = \{(y, x) | (x, y) \in U\}$.

   The elements of Φ are called *entourages*. If $x \in X$ and $U \in \Phi$, set $U[x] = \{y | (x, y) \in U\}$. A uniformity induces a topology on X: a set O is open if for every $x \in O$ there is an entourage V such that $V[x] \subseteq O$. A topological space is *uniformizable* if there is a uniformity compatible with the topology (that is, the topology generated by the uniformity is the topology on X).

   Prove that a metric space X is uniformizable.

2. Let T be the 2-dimensional torus. For each of the following, give an example or prove such an object does not exist.
   (1) A map $f : T \to S^2$ of degree 1.
   (2) A map $g : S^2 \to T$ of degree 1.

3. The torus T bounds a compact region R. Two copies of R are glued together by identifying their boundaries via the map $f : T \to T$ given by $f(\theta, \varphi) = (\theta + \varphi, \theta + 2\varphi)$ (here $T = S^1 \times S^1$, where points on S^1 are identified with an angle $\theta \in [0, 2\pi)$). The result is a compact 3-manifold M.
   (1) Compute the map $f_* : H_1(T) \to H_1(T)$.
   (2) Use the Mayer-Vietoris sequence to compute the homology of M.

4. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

5. Compute $H^\bullet(M; \mathbb{Z})$, where M is the space with homology groups

$$H_k(M; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0, 2, 4 \\ \mathbb{Z}_3 & k = 1 \\ \mathbb{Z}_2 & k = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Answer the following with complete definitions, statements, or short proofs.

6. Prove that for a finite CW-complex X, $H^1(X;\mathbb{Z})$ is free.

7. Compute $\chi(\mathbb{C}P^2 \times \mathbb{R}P^4 \times S^2)$

8. Give an example of a space that is connected but not path-connected.

9. State the Urysohn Metrization Theorem.

10. Prove that if $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

11. Let X be a space and let X^* be its one-point compactification. Prove that if X is locally compact and Hausdorff, then X^* is Hausdorff.

12. State the Lefschetz Fixed Point Theorem.

13. Describe all the connected covering spaces $E \to S^1 \times S^1$.

14. Does the following exact sequence of abelian groups necessarily split? Prove or give a counterexample.

$$0 \to A \to B \to \mathbb{Z} \to 0$$

15. Compute the integral homology of the space $\mathbb{R}P^2 \times \mathbb{C}P^3$.