Ph.D. Exam: Numerical Analysis, January, 2022.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Prove or provide a counterexample to the following statements. Justify each nontrivial step.

(a) The Frobenius norm satisfies $\|AB\|_F \leq \|A\|_F\|B\|_F$.

(b) $\|A\|_2 \leq \|A\|_F$, where $\|A\|_F$ is the Frobenius norm of A.

2. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{pmatrix}.$$

(a) Find an economy singular value decomposition of A.

(b) Find an economy QR decomposition of B.

3. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

(a) If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

4. Let $P \in \mathbb{C}^{m\times m}$ be a projector. Show that $\|P\|_2 = 1$ if and only if P is an orthogonal projector.

5. Suppose A is Hermitian positive definite.

(a) Prove that each principal submatrix of A is Hermitian positive definite.

(b) Prove that an element of A with largest magnitude lies on the diagonal.

(c) Prove that A has a Cholesky decomposition.

6. Consider $I = \int_{-1}^{1} |x| dx = 1$. Let $T(h)$ be the composite trapezoidal rule approximation to I, for $h = 2/n, n \in \mathbb{N}$.

(a) Show $T(2/n) = 1$ whenever n is even.

(b) Find $T(2/n)$ when n is odd, and determine the order of convergence in terms of h.

7. **(a)** Let $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$. Determine (explicitly find) an $\mathcal{O}(h^2)$ difference approximation to $f'(x_0)$ based on $f(x_0), f(x_1)$ and $f(x_2)$.

(b) Prove your solution to (a) is $\mathcal{O}(h^2)$

8. **(a)** Consider the inner product on $C(0,2)$ given by $(f,g) = \int_0^2 f(t)g(t)\,\mathrm{d}\,t$. Find three orthonormal polynomials ϕ_0, ϕ_1, ϕ_2 on $(0,2)$ with respect to the given inner product such that the degree of ϕ_n is equal to n, $n = 0, 1, 2$.

(b) Find the nodes t_1 and t_2 and weights w_1 and w_2 which yield the weighted Gaussian Quadrature formula

$$\int_0^2 f(t)\,\mathrm{d}\,t \approx w_1 f(t_1) + w_2 f(t_2)$$

with degree of exactness $m = 3$. **You should find the nodes exactly, and may leave the weights w_1, w_2 in integral form**.

9. Let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct numbers. Let $l_j(x)$ be the associated Lagrange basis polynomials, $j = 0, \ldots n$.

(a) State the definition of $l_j(x)$ and show that $\{l_j(x)\}_{j=0}^{n}$ form a basis for $\mathcal{P}_n$, the space of polynomials of degree at most n.

(b) Show that

$$\sum_{j=0}^{n} (x - x_j)^k l_j(x) = 0, \qquad \text{for all } \; k = 1, \ldots, n.$$

10. The *Littlewood-Salem-Izumi constant* α_0 is the unique solution $\alpha_0 \in (0,1)$ of

$$\int_0^{3\pi/2} \frac{\cos(t)}{t^{\alpha}}\,\mathrm{d}\,t = 0.$$

Describe how you could use Newton's method together with a composite Clenshaw-Curtis quadrature rule based on the $\mathcal{P}_1$ interpolant over n subintervals to approximate α_0. Be specific about how each function used in the Newton method is defined, and be specific about how the nodes and weights for the quadrature rule are defined.

For reference, the first few Chebyshev polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$.