Ph.D. Exam: Numerical Analysis, May, 2025
Do 4 (four) of the first 5 (1-5) and 4 (four) of the last 5 problems (6-10).

1. Assume $A \in C^{m\times m}$.
(a) Show that A has a Schur decomposition.
(b) If A is normal, show that A is diagonalizable.

2. Suppose A is Hermitian positive definite.
(a) Prove that each principal submatrix of A is Hermitian positive definite.
(b) Prove that an element of A with largest magnitude lies on the diagonal.
(c) Prove that A has a Cholesky decomposition.

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$ for all $x \in R^n$. That is to find C and c such that $c\|x\|_\infty \le \|x\|_2 \le C\|x\|_\infty$, for all $x \in R^n$. Note that the constants should be determined so that the equalities hold for some nonzero $x \in R^n$.
(b) Show that $\|QA\|_2 = \|A\|_2$ if Q is a unitary matrix.

4. Assume that $A \in C^{n\times n}$ and there exists $p \ge 1$ such that $\|A\|_p < 1$, where $\|\cdot\|_p$ is a vector-induced matrix norm.
(a) Prove that $I - A$ is invertible.
(b) Prove that

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(c) Prove that $\|A\|_q\|A^{-1}\|_q \ge 1$, $\forall 1 \le q \le \infty$.
(d) Prove that

$$\frac{1}{1 + \|A\|_p} \le \|(I - A)^{-1}\|_p \le \frac{1}{1 - \|A\|_p}.$$

5. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in C^{m\times n}$. Let u_j denote column j of U.

(a) Suppose rank(A)=$p < n < m$. Show $\{u_1, u_2, \cdots, u_p\}$ is a basis for $Col(A)$ and $\{u_{p+1}, u_{p+2}, \cdots, u_m\}$ is a basis for $Null(A^*)$.

(b) Suppose A is full rank and $x \neq 0$. Let σ_i, $i = 1, \cdots, n$ be the singular values of A. Show

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that it holds.

6. Consider the function $g(x) = e^{-x}$.

(a) Prove that g is a contraction on $G = [\ln 1.1, \ln 3]$.

(b) Prove that g maps $G = [\ln 1.1, \ln 3]$ into $G = [\ln 1.1, \ln 3]$.

(c) Prove that $x_{k+1} = g(x_k)$ converges to an unique fixed point $z \in G = [\ln 1.1, \ln 3]$ for any initial value $x_0 \in G = [\ln 1.1, \ln 3]$.

7. Based on $u_1(x) = 1, u_2(x) = x, u_3(x) = x^2$, use Gram-Schmidt orthogonalization process to compute the three polynomials $w_1(x), w_2(x), w_3(x)$ which are orthonormal on the interval $[0, 1]$ with respect to the inner product $(f, g) = \int_0^1 f(x)g(x)dx$. Using these polynomials, find the best approximation in $P^2[0, 1]$ for $f(x) = x^{\frac{1}{2}}$.

8. Consider the finite difference formula

$$f'(t_j) = \frac{1}{12h}\left[f(t_j - 2h) - 8f(t_j - h) + 8f(t_j + h) - f(t_j + 2h)\right] + O(h^4).$$

(a) Derive this formula by using Taylor's theorem.

(b) Derive this formula by using Lagrange polynomial representation.

9. Assume the numerical quadrature for $\hat{f}(\hat{x})$ on $[0, 1]$ is

$$\hat{J}(\hat{f}) = \int_0^1 \hat{f}(\hat{x})d\hat{x} \approx \hat{Q}(\hat{f}) = \sum_{j=0}^{m} \hat{\alpha}_j \hat{f}(\hat{x}_j).$$

Derive the numerical quadrature of $J(f) = \int_a^b f(x)\,dx$.

10. Consider numerically solving the initial value problem

$$y'(t) = f(t, y),\ 0 < t \leq t_f, \quad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that all convergent members of the family of methods

$$y_{n+2} + (\theta - 2)y_{n+1} + (1 - \theta)y_n = \frac{1}{4}h[(6 + \theta)f_{n+2} + 3(\theta - 2)f_n],$$

parameterized by θ, are also A_0-stable.