Ph.D. Exam: Numerical Analysis, May, 2026.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Prove or provide a counterexample for each of the following.

(a) If B is any submatrix of $A \in \mathbb{C}^{m\times n}$ then $\|B\|_p \le \|A\|_p$ for any matrix p-norm.

(b) If matrix $A \in \mathbb{C}^{n\times n}$ is invertible then $f_A(x,y) := x^*Ay$ is an inner product on $\mathbb{C}^n$.

2. Prove or provide a counterexample for each of the following.

(a) The Frobenius norm is invariant under unitary transformation ($\|UA\|_F = \|A\|_F$ and $\|AV\|_F = \|A\|_F$, for unitary matrices U, V).

(b) If $E = uv^*$ for vectors $u, v \in \mathbb{C}^n$, then $\|E\|_2 = \|u\|_2\|v\|_2$.

3. Prove or provide a counterexample for each of the following.

(a) Any square matrix A has a decomposition Q^*TQ where Q is unitary and T is triangular.

(b) The spectral radius is equal to the matrix 2-norm for any square matrix A.

4. **(a)** Let $v = (1,0,1,0,1)^T$ and $x = (1,1,2,3,6)^T$. Find $w \in \mathbb{C}^5$ and $c \in \mathbb{C}$ such that $x = cv + w$ and $w^*v = 0$. Is there any other vector w and/or scalar c that will work? Explain.

(b) Compute a full singular value decomposition (SVD) of the following matrix, and comment on its uniqueness.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

5. **(a)** Suppose A is $n\times n$ and nonsingular, and exact data b and solution x satisfy $Ax = b$. Suppose data pertubation Δb and solution perturbation Δx further satisfy $A(x + \Delta x) = (b + \Delta b)$. You may assume $\|\cdot\|$ is an induced vector norm. Show

$$\frac{\|\Delta x\|}{\|x\|} \le \operatorname{cond}(A)\frac{\|\Delta b\|}{\|b\|}.$$

(b) Let $A \in \mathbb{C}^{m\times n}$ with $m > n$. Suppose A is full rank, and consider the least-squares problem of finding $x \in \mathbb{C}^n$ that minimizes $\|Ax - b\|$ in the 2-norm. Describe how to solve the least-squares problem using the QR decomposition of A, and explain why this is preferable to solving the normal equations directly.

- For the problems that follow, $\mathcal{P}_k$ denotes the space of polynomials of degree at most k.
- The recursion for the Chebyshev polynomials of the first kind is given by $T_0(x) = 1$, $T_1(x) = x$, and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

6.

(a) Find the values of α and β that minimize

$$\int_0^1 (x^5 - (1 + \alpha x + \beta x^2))^2 \, \mathrm{d}\, x.$$

(b) Find the error in the best uniform approximation of x^6 over $[-1, 1]$ by $p(x) \in \mathcal{P}_5$. Justify your solution.

7. Prove for any $f \in C[a, b]$ and integer $n \geq 0$, that the best uniform approximation of f in $\mathcal{P}_n$ is unique. You may assume the existence of at least one best uniform approximation of f.

8. Suppose $f \in C^{n+1}[a, b]$ and $p(x) \in \mathcal{P}_n$ interpolates f at $a \leq x_0 \leq x_1 < \ldots < x_n \leq b$. Consider an arbitrary fixed $\hat{x} \in \{x_0, x_1, \ldots, x_n\}$, and derive an exact expression for the error in the first derivative at $\hat{x}$, $f'(\hat{x}) - p'(\hat{x})$.

9. Consider using Newton's method to find $x \in \mathbb{R}^n$ such that $F(x) = 0$, for $F : D \subseteq R^n \to \mathbb{R}^n$, where D is open and convex and F is continuously differentiable on D.

(a) Write down the Newton iteration for finding x_{k+1} from x_k.

(b) Suppose $\bar{x}$ satisfies $F(\bar{x}) = 0$; and $J(x)$, the Jacobian of F evaluated at x satisfies $\|J^{-1}(x)\| \leq \mu$ for some number $\mu > 0$ for all x in a neighborhood $N \subseteq D$ of $\bar{x}$. Show for $x_k \in N$ that

$$\|x_{k+1} - \bar{x}\| \leq \mu \|\bar{x} - x_k\| \cdot \max_{0 \leq t \leq 1} \|J(x_k + t(\bar{x} - x_k)) - J(x_k)\|.$$

10. Consider the finite difference approximation to the first derivative based on $p_1 \in \mathcal{P}_1$ given by $f'(a) \approx p_1'(a) = \frac{f(a+h) - f(a)}{h}$. Define

$$\psi(h) = \frac{f(a + h) - f(a)}{h} = \psi(0) + a_1 h + a_2 h^2 + a_3 h^3 + \ldots.$$

with $\psi(0) = \lim_{h \to 0} \psi(h) = f'(a)$, at least in exact arithmetic. Let $\psi_k(h)$ be the result of k Richardson extrapolations, *e.g.*, $\psi_1(h) = 2\psi(h/2) - \psi(h)$.

(a) Find and prove inductively the general formula for $\psi_k(h)$ based on $\psi_{k-1}(h)$ and $\psi_{k-1}(h/2)$.

(b) State two possible advantages and two possible disadvantages of using Richardson extrapolation.