Ph.D. Exam: Numerical Analysis, January, 2026.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Prove each of the following.

(a) The matrix 2-norm is invariant under unitary transformation.

(b) If $n \times n$ matrix A is diagonal, then for any matrix p-norm $\|\cdot\|_p$, it holds that $\|A\|_p = \max_{1\leq i\leq n} |a_{ii}|$.

2. Suppose $A \in \mathbb{C}^{m\times n}$. Prove or provide a counterexample for each of the following.

(a) The Frobenius norm of a matrix is induced by (subordinate to) a vector norm.

(b) $\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$.

3. Let $A \in \mathbb{C}^{n\times n}$. You may use the fact (if you find it useful) That A has a Schur decomposition. Prove or provide a counterexample for each of the following.

(a) If A is both normal and triangular then it is diagonal.

(b) If A is normal then the 2-norm of A is equal to it's spectral radius $\rho(A)$.

4. **(a)** Let $r_i = \sum_{j=1,j\neq i}^{n} |a_{ij}|$. Let D_i be the disk in $\mathbb{C}$ with center a_{ii} and radius r_i. Prove that if λ is an eigenvalue of A, then $\lambda \in \bigcup_i D_i$; in other words, λ is in at least one of the disks D_i.

(b) Determine all eigenvalues of the Householder reflector $H(w)$, including their multiplicities. Justify your answer.

5. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a full singular value decomposition of A.

(b) Find an economy QR decomposition of B.

6. Consider the function $g(x) = e^{-x}$.

(a) Prove that g is a contraction on $G = [\ln 1.1, \ln 3]$.

(b) Prove that g maps $G = [\ln 1.1, \ln 3]$ into $G = [\ln 1.1, \ln 3]$.

(c) Prove that $x_{k+1} = g(x_k)$ converges to an unique fixed point $z \in G = [\ln 1.1, \ln 3]$ for any initial value $x_0 \in G = [\ln 1.1, \ln 3]$.

7. Consider the finite difference formula

$$f'(t_j) = \frac{1}{12h}\left[f(t_j - 2h) - 8f(t_j - h) + 8f(t_j + h) - f(t_j + 2h)\right] + O(h^4).$$

(a) Derive this formula by using Taylor's theorem.

(b) Derive this formula by using Lagrange polynomial representation.

8. Given a differentiable function $f(x)$, consider the problem of finding a polynomial $p(x) \in \mathbb{P}^n$ such that

$$p(x_0) = f(x_0), \quad p'(x_i) = f'(x_i), \quad i = 1, 2, \cdots, n,$$

where $x_i, i = 1, 2, \cdots, n$, are distinct nodes. (It is not excluded that $x_1 = x_0$.) Show that the problem has a unique solution and describe how it can be obtained.

9. Let $Q_m(f)$ denote the m-point Gaussian quadrature rule over the interval $[a, b]$ and with continuous weight function $\rho(x) \geq 0$, that is,

$$Q_m(f) = \sum_{i=1}^{m} \alpha_i f(x_i) \approx J(f) = \int_a^b \rho(x) f(x)\, dx.$$

Show that, if a and b are finite and f is continuous, then $Q_m(f) \to J(f)$ as $m \to \infty$.

10. Consider numerically solving the initial value problem

$$y'(t) = f(t, y), \ 0 < t \leq t_f, \quad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that all convergent members of the family of methods

$$y_{n+2} + (\theta - 2)y_{n+1} + (1 - \theta)y_n = \frac{1}{4}h[(6 + \theta)f_{n+2} + 3(\theta - 2)f_n],$$

parameterized by θ, are also A_0-stable.