Ph.D. Exam: Numerical Analysis, January, 2024
Do 4 (four) of the first 5 (1-5) and 4 (four) of the last 5 problems (6-10).

1. Assume $A \in C^{m \times m}$.
(a) Show that A has a Schur decomposition.
(b) If A has a collection of m linearly dependent eigenvectors, show that A is diagonalizable.

2. Let matrix $A \in \mathcal{C}^{m \times n}$ with $n < m$. Let $b \in \mathcal{C}^m$, and let r denote the residual vector $r = b - Ax$.
(a) Show that x solves the least squared problem $\min \|b - Ax\|_2$ if and only if $r \in Null(A^*)$.
(b) Suppose A is full rank, and describe how to find the least squares solution using the QR decomposition of A.

3. Let $A \in C^{m \times n}$, with $m \geq n$ and $rank(A) = p = n \geq 3$. Let $a_1, a_2, \cdots$ denote the columns of A.
(a) Using the modified Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.
(b) Show the vector q_3 found in part (a) is orthogonal to both q_1 and q_2.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.
(a) If E is a $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is a $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

5. If $q_1, \cdots, q_n$ is an orthonormal basis for the subspace $V \subset C^{m \times n}$ with $m > n$, prove that the orthogonal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.

6. Derive the three-point formula for the second derivative

$$f''(x) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12} f^{(4)}(\eta),$$

for some $\eta \in [x_0 - h, x_0 + h]$.

7. Consider

$$y' = 4ty; \qquad t \in [0, 1]; \qquad y(0) = 2, \tag{1}$$

which has solution $Y(t) = 2e^{2t^2}$.

(a) Derive an error bound for the forward Euler scheme.

(b) Derive the Taylor method of order 2 for (1).

8. Let $\mathcal{P}_1$ be the space of polynomials of degree at most one. Using the norm $\|u\|_2 = \left(\int_a^b u^2 dx \right)^{\frac{1}{2}}$.

(a) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a, b] = [-1, 1]$.

(b) Find the least-squares approximation to $f(x) = x^4$ in $\mathcal{P}_1$ over $[a, b] = [0, 1]$.

9. Consider the fixed point problem $x = f(x)$ where $f(x) = e^{-(3+x)}$.

(a) Assuming all computations are done in exact arithmetic, find the largest open interval in R where the fixed point iteration $x_{k+1} = f(x_k)$ is ensured to converge.

(b) Write a Newton iteration for finding the fixed point.

10. Suppose $f \in C^{n+1}[a, b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i))\}_{i=0}^n$, where $x_0, \cdots, x_n$, are distinct points in $[a, b]$. Consider an arbitrary fixed $x \in [a, b]$, and derive an exact expression for the error $f(x) - p(x)$.