Ph.D. Exam: Numerical Analysis, May, 2022.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Prove or provide a counterexample to each of the following.

(a) If matrix A is normal and triangular, then it is diagonal.

(b) Every matrix has a Schur factorization.

2. **(a)** For $v \in \mathbb{C}^n$, define $f_A(v) := (v^* A v)^{1/2}$. Under what condition(s) on $A \in \mathbb{C}^{n \times n}$ is $f_A(\cdot)$ a norm on $\mathbb{C}^n$? (*Full credit will only be given for a general condition or conditions. An example is not sufficient.*) Prove that $f_A(\cdot)$ is a norm on $\mathbb{C}^n$ under the condition(s) you require.

(b) Assuming the condition(s) you require in **(a)**, given a matrix A, determine the best constants α and β to satisfy the inequality

$$\alpha \|v\|_2 \leq f_A(v) \leq \beta \|v\|_2.$$

3. Let $A = U \Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m \times n}$. Let u_j denote column j of U.

(a) Suppose $\operatorname{rank}(A) = p < n < m$. Show $\{u_1, u_2, \ldots, u_p\}$, is a basis for $\operatorname{Col}(A)$, and $\{u_{p+1}, u_{p+2}, \ldots, u_m\}$ is a basis for $\operatorname{Null}(A^*)$.

(b) Suppose A is full rank and $x \neq 0$. Let $\sigma_i,\ i = 1, \ldots, n$, be the singular values of A. Show

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that also.

4. Let $\{a_1, \ldots, a_n\}$ be a linearly independent set of vectors. Consider the Gram-Schmidt and modified Gram-Schmidt algorithms for computing an orthonormal basis $\{q_1, \ldots, q_n\}$ so that $\operatorname{span}\{q_1, \ldots q_k\} = \operatorname{span}\{a_1, \ldots a_k\}$, for each $k = 1, \ldots, n$.

Suppose in computing q_2, an orthogonalization error is committed so that $q_2^* q_1 = \varepsilon$.

(a) Use the Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

(b) Use the modified Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

5. Compute the Cholesky decomposition of the following matrix, or explain why it does not exist.

$$A = \begin{pmatrix} 1 & 1/2 & 2 & 3 \\ 1/2 & 5/16 & 3/2 & 5/2 \\ 2 & 3/2 & 17 & 17 \\ 3 & 5/2 & 17 & 31 \end{pmatrix}.$$

6. **(a)** Let $G = [0, 2]$ and

$$g(x) = \frac{1}{3}(\frac{x^3}{3} - x^2 - \frac{5}{4}x + 4).$$

Use the contraction mapping theorem to prove that if $x_0 \in G$, then the sequence defined by $x_{k+1} = g(x_k), (k = 0, 1, \cdots)$ converges to a unique fixed point $z \in G$.

(b) Consider the fixed point iteration method $x_{k+1} = g(x_k)$, $k = 0, 1, \cdots$ for solving the nonlinear equation $f(x) = 0$. Consider choosing an iteration function of the form

$$g(x) = x - af(x) - b(f(x))^2 - c(f(x))^3,$$

where a, b, and c are parameters to be determined. Assume f is sufficiently differentiable and the corresponding iterations $x_{k+1} = g(x_k)$ converge to a unique fixed point z. Find expressions for the parameters a, b, and c in terms of functions of z such that the iteration method is of fourth order.

7. Consider $f(t) = \sin(t)$.

(a) Without using orthogonal polynomials, find the best approximation $p_1(t) \in P^2[-1, 1]$ to $f(t) \in C[-1, 1]$ with respect to the L^2 norm.

(b) Find the Taylor polynomial approximation $p_2(t)$ of degree 3 at $t = 0$.

(c) Find the Lagrange polynomial approximation $p_3(t)$ of degree 3 that interpolates $f(t)$ at $t = -1, -\frac{1}{3}, \frac{1}{3}, 1$.

8. Given a differentiable function $f(x)$, consider the problem of finding a polynomial $p(x) \in \mathbb{P}^n$ such that

$$p(x_0) = f(x_0), \quad p'(x_i) = f'(x_i), \quad i = 1, 2, \cdots, n,$$

where $x_i, i = 1, 2, \cdots, n$, are distinct nodes. (It is not excluded that $x_1 = x_0$.) Show that the problem has a unique solution and describe how it can be obtained.

9. Let $Q_m(f)$ denote the m-point Gaussian quadrature rule over the interval $[a, b]$ and with continuous weight function $\rho(x) \geq 0$, that is,

$$Q_m(f) = \sum_{i=1}^{m} \alpha_i f(x_i) \approx J(f) = \int_a^b \rho(x) f(x)\, dx.$$

Show that, if a and b are finite and f is continuous, then $Q_m(f) \to J(f)$ as $m \to \infty$.

10. Consider numerically solving the initial value problem

$$y'(t) = f(t, y), \ 0 < t \leq t_f, \quad \text{with } y(0) = \eta.$$

Assume f is sufficiently differentiable and let h denote the step size. Show that all convergent members of the family of methods

$$y_{n+2} + (\theta - 2)y_{n+1} + (1 - \theta)y_n = \frac{1}{4}h[(6 + \theta)f_{n+2} + 3(\theta - 2)f_n],$$

parameterized by θ, are also A_0-stable.