Ph.D. Exam: Numerical Analysis, August, 2021.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

You may use the following degree n orthogonal polynomials as you like.

n	Legendre	Chebyshev
0	1	1
1	x	x
2	$3x^2 - 1$	$2x^2 - 1$
3	$5x^3 - 3x$	$4x^3 - 3x$
4	$35x^4 - 30x^2 + 3$	$8x^4 - 8x^2 + 1$

1. Suppose $A \in \mathbb{C}^{m\times m}$.

(a) Suppose A is normal and upper-triangular. Show that A is diagonal.

(b) Prove that a Schur decomposition diagonalizes A if and only if A is normal.

2. **(a)** Suppose $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

(b) Suppose $A \in \mathbb{C}^{m\times m}$, and let a_j be the jth column of A. Prove the inequality

$$|\det A| \leq \prod_{j=1}^{m} \|a_j\|_2. \quad (\textit{Hint: it may be useful to consider the QR decomposition of } A).$$

3. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}.$$

(a) Find an economy singular value decomposition of A.

(b) Find an economy QR decomposition of B.

4. Let $\{a_1, \ldots, a_n\}$ be a linearly independent set of vectors. Consider the Gram-Schmidt and modified Gram-Schmidt algorithms for computing an orthonormal basis $\{q_1, \ldots, q_n\}$ so that $\text{span}\{q_1, \ldots q_k\} = \text{span}\{a_1, \ldots a_k\}$, for each $k = 1, \ldots, n$.

Suppose in computing q_2, an orthogonalization error is committed so that $q_2^* q_1 = \varepsilon$.

(a) Use the Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

(b) Use the modified Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

5. Suppose $A \in \mathbb{C}^{m\times m}$.

(a) Prove Gerschgorin's disk theorem: Let $r_i = \sum_{j=1, j\neq i}^{m} |a_{ij}|$. Let D_i be the disk in $\mathbb{C}$ with center a_{ii} and radius r_i. If λ is an eigenvalue of A, then $\lambda \in \bigcup_i D_i$; in other words, λ is in at least one of the disks D_i.

(b) Suppose A is Hermitian positive definite. Prove that an element of A with largest magnitude lies on the diagonal.

6. Consider the function $f(t)$ on $[-1,1]$ given by

$$f(t) = \begin{cases} 1+t, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1. \end{cases}$$

Let $\mathcal{P}_k$ be the space of polynomials of degree at most k.

(a) Find the best uniform approximation of f in $\mathcal{P}_1$ over $[-1,1]$.

(b) Find the best approximation of f in $\mathcal{P}_1$ in the norm induced by the inner-product $(u,v) = \int_{-1}^{1} uv \, \mathrm{d}t$.

(c) Find the best approximation of f in $\mathcal{P}_2$ in the norm induced by the inner-product $(u,v) = \int_{-1}^{1} uv \, \mathrm{d}t$.

7. Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$, with respect to the inner-product $(f,g) = \int_a^b f(x)g(x)w(x) \, \mathrm{d}x$, indexed so that ϕ_k is of degree k. Recall that the weight function w for the inner-product satisfies $w \in L^\infty$ and $w(x) > 0$ for almost every $x \in [a,b]$. Prove that ϕ_k has k distinct roots $\{x_j^{(k)}\}_{j=1}^{k}$, with $x_j^{(k)} \in [a,b]$, $j = 1, \ldots, k$.

8. Find $x_0, x_1, x_2 \in [a,b]$, and $c_0, c_1, c_2 \in \mathbb{R}$, that maximize the degree of exactness for the quadrature formula

$$\int_a^b f(x) \, \mathrm{d}x \approx c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2).$$

9. The *Littlewood-Salem-Izumi constant* α_0 is the unique solution $\alpha_0 \in (0,1)$ of

$$\int_0^{3\pi/2} \frac{\cos(t)}{t^\alpha} \, \mathrm{d}t = 0.$$

Describe how you could use Newton's method together with a composite 2 point Gauss quadrature rule over N subintervals to approximate α_0. Be specific about how each function used in the Newton method is defined, and be specific about how the nodes and weights for the quadrature rule are defined.

10. Consider the ODE $y' = f(x,y)$ on an interval $a \le x \le b$, with $h = (b-a)/N$ for some given value of N.

(a) Suppose f is C^1 on $[a,b]$. Compute the local truncation error (also called truncation error) for the forward Euler method $y_{k+1} = y_k + hf(x_k, y_k)$.

(b) Suppose $f = y^\lambda$ and $y(a) = 1$. For what values of λ will the forward Euler method converge for h small enough? You are not required to state how small h needs to be. If your solution requires any additional assumptions, they should be clearly stated.