Ph.D. Exam: Numerical Analysis, May, 2021.
Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$. Let u_j denote column j of U.

(a) Suppose $\text{rank}(A) = p < n < m$. Show $\{u_1, u_2, \dots, u_p\}$, is a basis for $\text{Col}(A)$, and $\{u_{p+1}, u_{p+2}, \dots, u_m\}$ is a basis for $\text{Null}(A^*)$.

(b) Suppose $k < \text{rank}(A)$. What is the least-squares solution x that minimizes $\|b - Cx\|_2$, where C is the rank-k matrix that minimizes $\|A - C\|_2$?

2. For $x, y > 0$, consider computing $f(x,y) = \sqrt{y + x^2} - \sqrt{y}$ in floating-point arithmetic with machine precision ε_m.

(a) Explain the difficulties in computing $f(x,y)$, if $x^2 \ll y$. What are the absolute and relative errors if $x^2/y < \varepsilon_m$, if $f(x,y)$ is computed directly from the form given above?

(b) Suppose $x^2/y < \varepsilon_m$. Describe a way to compute $f(x,y)$ with more accuracy in this situation.

3. Let $\{a_1, \dots, a_n\}$ be a linearly independent set of vectors. Consider the Gram-Schmidt and modified Gram-Schmidt algorithms for computing an orthonormal basis $\{q_1, \dots, q_n\}$ so that $\text{span}\{q_1, \dots q_k\} = \text{span}\{a_1, \dots a_k\}$, for each $k = 1, \dots, n$.

Suppose in computing q_2, an orthogonalization error is committed so that $q_2^* q_1 = \varepsilon$.

(a) Use the Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

(b) Use the modified Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

4. Compute the Cholesky decomposition of the following matrix, or explain why it does not exist.

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 5 & 4 & 2 \\ 2 & 4 & 14 & 1 \\ 0 & 2 & 1 & 5 \end{pmatrix}.$$

5. Suppose A is Hermitian positive definite.

(a) Prove that each principal submatrix of A is Hermitian positive definite.

(b) Prove that an element of A with largest magnitude lies on the diagonal.

6. Show that if a polynomial of degree at most n has $n+1$ (or more) zeros, then it is the zero polynomial.

7. **(a)** Prove for any $f \in C[a,b]$ and integer $n \geq 0$, that the best uniform approximation of f in P_n is unique. You may assume the existence of at least one best uniform approximation of f.

(b) Is there some $a \in \mathbb{R}$ for which $p_1(x) = ax + 1$ is the best uniform approximation to x^3 in P_1 over $[-1,1]$? If so, what is it? EXPLAIN.

8. Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$, with respect to the inner-product $(f,g) = \int_a^b f(x)g(x)w(x)\,\mathrm{d}\,x$, indexed so that ϕ_k is of degree k. Let $\mathcal{P}_k$ be the space of polynomials of degree at most k.

(a) Determine the weights w_j so that the quadrature rule $I_n(f) = \sum_{j=0}^{n} w_j f(x_j)$ satisfies $I_n(f) = (1,f)$ for any $f \in \mathcal{P}_n$, so long as the points $\{x_j\}_{j=0}^{n}$ are distinct. You may present the weights as a set of linear equations (which you are not required to solve), or as a set of integrals (which you are not required to integrate). Justify your solution.

(b) Prove that if the interpolation points $\{x_j\}_{j=0}^{n}$ are the zeros of ϕ_{n+1}, then $I_n(f) = (1,f)$ for any $f \in \mathcal{P}_{2n+1}$.

9. Suppose $f \in C^{n+1}[a,b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i)\}_{i=0}^{n}$, where $x_0, \ldots, x_n$, are distinct points in $[a,b]$. Consider an arbitrary fixed $x \in [a,b]$, and derive an exact expression for the error $f(x) - p(x)$.

10. The *Littlewood-Salem-Izumi constant* α_0 is the unique solution $\alpha_0 \in (0,1)$ of

$$\int_0^{3\pi/2} \frac{\cos(t)}{t^\alpha}\,\mathrm{d}\,t = 0.$$

Describe how you could use Newton's method together with a composite Clenshaw-Curtis quadrature rule based on the $\mathcal{P}_1$ interpolant over n subintervals to approximate α_0. Be specific about how each function used in the Newton method is defined, and be specific about how the nodes and weights for the quadrature rule are defined.

For reference, the first few Chebyshev polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$.