Ph.D. Exam: Numerical Analysis, August, 2020.

Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$ with $\text{rank}\,(A) = p$ and $p \leq n \leq m$.

(a) Show $\text{Col}\,(A^*) = \text{Span}\,\{v_1, v_2, \ldots, v_p\}$, where $v_1, \ldots, v_p$ are the first p columns of V.

(b) Show $\text{Null}\,(A) = \text{Span}\,\{v_{p+1}, v_{p+2}, \ldots, v_n\}$.

(c) Find the economy singular value decompositions of the matrix A given by

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 4 \end{pmatrix}.$$

2. Let $A \in \mathbb{C}^{m\times n}$ with $\text{rank}(A) = n < m$. Let $A = QR$ be the QR decomposition of A, and $A = Q_1R_1$ be the economy QR decomposition.

(a) Show that $x \in \mathbb{C}^n$ that solves $R_1x = Q_1^*b$ is the least-squares solution to $Ax = b$ for $b \in \mathbb{C}^m$.

(b) Find the economy QR decomposition of matrix B given by

$$B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Let A and S be $n \times n$ matrices, and suppose S is nonsingular.

(a) Show that A and $B = SAS^{-1}$ have the same eigenvalues, and provide the relation between the eigenvectors of B and the eigenvectors of A.

(b) Suppose A is skew-Hermitian. Show the eigenvalues of A are pure imaginary.

4. Suppose the 5×5 symmetric matrix A has eigenvalues known to within the given tolerances.

$$\begin{aligned} 3.5 > \lambda_1 &> 2.5 \\ 2.0 > \lambda_2 &> 1.0 \\ 1.0 > \lambda_3 &> -1.0 \\ -1.0 > \lambda_4 &> -2.0 \\ -2.5 > \lambda_5 &> -3.5. \end{aligned}$$

(a) Describe how shifting can be used so that the power method can be used to compute λ_1 with guaranteed convergence.

(b) What is the shift that provides the best convergence rate in the worst case?

5. For $x, y > 0$, consider computing $f(x,y) = \sqrt{y+x^2} - \sqrt{y}$ in floating-point arithmetic with machine precision ϵ_m.

(a) Explain the difficulties in computing $f(x,y)$, if $x \ll y$. What are the absolute and relative errors if $x^2/y < \epsilon_m$, if $f(x,y)$ is computed directly from the form given above?

(b) Suppose $x^2/y < \epsilon_m$. Describe a way to compute $f(x,y)$ with more accuracy in this situation.

6. Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$, with respect to the inner-product $(f,g) = \int_a^b f(x)g(x)w(x)\,\mathrm{d}\,x$, indexed so that ϕ_k is of degree k. Prove that ϕ_k has k distinct roots $\{x_j^{(k)}\}_{j=1}^{k}$, with $x_j^{(k)} \in [a,b]$, $j = 1, \ldots, k$.

7. Consider the interval $[a,b]$ with the partition $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. Suppose $s(x)$ is the natural cubic spline that interpolates the data $\{(x_i, y_i)\}_{i=1}^{n+1}$, and that $g \in C^2[a,b]$ interpolates the same data. Show that

$$\int_a^b (s''(x))^2\,\mathrm{d}\,x \le \int_a^b (g''(x))^2\,\mathrm{d}\,x.$$

8. **(a)** Consider the inner product on $C(0,2)$ given by $(f,g) = \int_0^2 f(t)g(t)\,\mathrm{d}\,t$. Find three orthonormal polynomials ϕ_0, ϕ_1, ϕ_2 on $(0,2)$ with respect to the given inner product such that the degree of ϕ_n is equal to n, $n = 0, 1, 2$.

(b) Find the nodes t_1 and t_2 and weights w_1 and w_2 which yield the weighted Gaussian Quadrature formula

$$\int_0^2 f(t)\,\mathrm{d}\,t \approx w_1 f(t_1) + w_2 f(t_2)$$

with degree of exactness $m = 3$. **You should find the nodes exactly, and may leave the weights w_1, w_2 in integral form.**

9. Prove for any $f \in C[a,b]$ and integer $n \ge 0$, that the best uniform approximation of f in P_n is unique. You may assume the existence of at least one best uniform approximation of f.

10. Suppose $f \in C^{n+1}[a,b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i)\}_{i=0}^{n}$, where $x_0, \ldots, x_n$, are distinct points in $[a,b]$. Consider an arbitrary fixed $x \in [a,b]$, and derive an exact expression for the error $f(x) - p(x)$.