University of Florida **August 2023**

Ph.D. Qualifying Exam in Probability

Carefully justify your answers

There are 7 problems

Problem 1.

Let $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. Prove that

$$\mathbb{P}(X + Y \geq 0) \leq \mathbb{P}(X \geq 0) + \mathbb{P}(Y \geq 0).$$

Problem 2.

1. Give the mathematical definition of:
   (a) convergence almost sure,
   (b) convergence in probability,
   (c) convergence in L^1,
   (d) convergence in distribution.

2. For $n \geq 1$, let X_n be uniformly distributed on $\{1, \ldots, n\}$, that is

$$\mathbb{P}(X_n = k) = \frac{1}{n}, \quad k \in \{1, \ldots, n\}.$$

   Prove that $\{\frac{X_n}{n}\}_{n\geq 1}$ converges to U in distribution, where U is uniform on $[0, 1]$.

Problem 3.

Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. continuous random variables having a uniform distribution on $[0, 1]$. Define, for $n \geq 1$,

$$Y_n = \max(X_1, \ldots, X_n).$$

1. Find the cumulative distribution function (CDF) of Y_n, $n \geq 1$.

2. Compute $\mathbb{E}[Y_n]$ and $\mathrm{Var}(Y_n)$.

3. Prove that $\{Y_n\}$ converges to 1 in L^1. What about almost sure convergence?

Problem 4. Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with a Poisson distribution of parameter λ.

1. Let $n \geq 1$. Without justification, what is the distribution of $X_1 + \cdots + X_n$?

2. State the weak and strong law of large numbers for i.i.d random variables.

3. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be an arbitrary bounded continuous function. Prove that

$$\lim_{n\to+\infty} \sum_{k=0}^{+\infty} e^{-n\lambda} \frac{(n\lambda)^k}{k!} \phi\left(\frac{k}{n}\right) = \phi(\lambda).$$

Problem 5.

1. Give the <u>mathematical</u> definition of standard Brownian motion.
2. Let $\{B_t\}$ be a standard Brownian motion. Prove that $\{\frac{B_n}{n}\}$ converges to 0 almost surely as $n \to +\infty$ ($n \in \mathbb{N}$).
3. Find $\mathbb{P}(B_2 \geq 0, B_1 \leq 0)$.

Problem 6.

Let $\{M_n\}_{n\geq 0}$ be a sequence of integrable random variables adapted to a filtration $\{\mathcal{F}_n\}$. Assume that for each bounded stopping time T, $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. Show that $\{M_n\}_{n\geq 0}$ is a martingale.

Problem 7.

Let $\{B_t\}$ be a standard Brownian motion. Define, for $x \in \mathbb{R}$,

$$T_x = \min\{t \geq 0 : B_t = x\}.$$

1. Prove that for all $x \in \mathbb{R}$, T_x is finite almost surely (that is, $\mathbb{P}(T_x < +\infty) = 1$).
2. Find $\mathbb{P}(T_3 < T_{-1})$. You can use a picture as justification.