**University of Florida** **May 2023**

Ph.D. Qualifying Exam in Probability

**Carefully justify your answers**

**There are 8 problems**

**Problem 1.**

Give the mathematical definition of:

1. conditional expectation,
2. martingale,
3. Poisson process,
4. standard Brownian motion.

**Problem 2.**

1. State the weak and strong law of large numbers for i.i.d random variables.
2. Prove the strong law of large numbers under the additional assumption of finite fourth moment ($\mathbb{E}[X_1^4] < +\infty$).
3. Let $\{B_t\}$ be a standard Brownian motion. Prove that $\{\frac{B_n}{n}\}$ converges to 0 almost surely as $n \to +\infty$ ($n \in \mathbb{N}$).

**Problem 3.**

1. Let $X$ be a random variable with a standard Gaussian distribution. Find the probability density function of $e^X$.
2. Let $X$ be a random variable with a standard Cauchy distribution. Find the probability density function of $\frac{1}{X}$.

**Problem 4.**

1. Let $p \geq 1$. Prove that if $X$ is a random variable, then

$$\mathbb{E}[|X|^p] = \int_0^{+\infty} p\, x^{p-1}\, \mathbb{P}(|X| \geq x)\, dx.$$

2. Let $p \geq 1$. Let $X$ and $Y$ be two independent random variables with $\mathbb{E}[Y] = 0$. Show that

$$\mathbb{E}[|X + Y|^p] \geq \mathbb{E}[|X|^p].$$

**Problem 5.**

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $0 < \sum_{k=1}^{+\infty} |a_k|^2 < +\infty$. Denote $\|a_n\| = \sqrt{\sum_{k=1}^{n} |a_k|^2}$ and $\|a\| = \sqrt{\sum_{k=1}^{+\infty} |a_k|^2}$.

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. symmetric Bernoulli, that is

$$\forall k \geq 1, \qquad \mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}.$$

Denote, for $n \geq 1$, $S_n = \sum_{k=1}^{n} a_k X_k$, and denote $\mathcal{F}_n = \sigma\{S_1, \dots, S_n\}$.

1. Prove that $\{S_n\}$ is an $\{\mathcal{F}_n\}$-martingale.
2. Prove that $\{S_n\}$ converges almost surely.
3. Prove that for all $n \geq 1$, for all $\lambda \in \mathbb{R}$,

   $$\mathbb{E}[e^{\lambda S_n}] \leq e^{\frac{\lambda^2 \|a_n\|^2}{2}}.$$

   (**Hint:** $\frac{e^x + e^{-x}}{2} \leq e^{\frac{x^2}{2}}$)
4. Use question 3. and the symmetry of $S_n$ to prove that for all $n \geq 1$, for all $x \geq 0$,

   $$\mathbb{P}(|S_n| \geq x) \leq 2\, e^{-\frac{x^2}{2\|a_n\|^2}}.$$

   (**Hint:** The minimum of the function $\lambda \mapsto e^{-\lambda x} e^{\frac{\lambda^2 \|a_n\|^2}{2}}$ is attained at $\lambda = \frac{x}{\|a_n\|^2}$)
5. Deduce the Khintchine inequality: $\forall p \geq 1$, $\forall n \geq 1$,

   $$\mathbb{E}\left[\left|\sum_{k=1}^{n} a_k X_k\right|^p\right]^{\frac{1}{p}} \leq C \|a_n\|,$$

   where $C$ is a numerical constant depending on $p$ only (You do not need to compute $C$).

   (**Hint:** $\mathbb{E}[|X|^p] = \int_0^{+\infty} p\, x^{p-1}\, \mathbb{P}(|X| \geq x)\, dx$)

**Problem 6.**

1. Give the definition of convergence of a sequence of random variables in probability and in distribution.
2. Which mode of convergence is stronger between convergence in probability and in distribution? Prove it.
3. Give a counterexample showing that convergence in probability is not equivalent to convergence in distribution.

**Problem 7.**

Let $\{B_t\}$ be a standard Brownian motion. Define, for $x \in \mathbb{R}$,

$$T_x = \min\{t \geq 0 : B_t = x\}.$$

1. Prove that for all $x \in \mathbb{R}$, $T_x$ is finite almost surely (that is, $\mathbb{P}(T_x < +\infty) = 1$).
2. Find $\mathbb{P}(T_{-2} < T_1)$. You can use a picture as justification.

**Problem 8.**

Let $n \in \mathbb{N}$. Let $X_1, \dots, X_n$ be i.i.d. random variables in $L_1$. Find $\mathbb{E}[X_1 | X_1 + \cdots + X_n]$.