University of Florida **August 2020**

Ph.D. Qualifying Exam in Probability

Carefully justify your answers

Problem 1.

1. State the weak and strong law of large numbers for i.i.d random variables.
2. Prove the weak law of large numbers (for i.i.d random variables).
3. Prove the strong law of large numbers under the additional assumption of finite fourth moment ($\mathbb{E}[X_1^4] < +\infty$).

Problem 2.
Let X, Y be independent and uniformly distributed on $[0,1]$.

1. Find the density of the random variable $X + Y$.
2. Find the density of the random variable $\mathbb{E}[X|X+Y]$.

Problem 3.
Let $\{X_n\}$ be a sequence of random variables uniformly bounded, that is, there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|X_n| \leq M$.
Prove that $\{X_n\}$ converges to 0 in L^1 if and only if $\{X_n\}$ converges to 0 in probability.

Problem 4.
Let $\{X_n\}$ be a sequence of random variables such that for all $n \geq 1$, X_n has a Poisson distribution of parameter n.
Does the random variable

$$\frac{X_n - n}{\sqrt{n}}$$

converges in distribution as $n \to +\infty$? If so, what is the limit? What about convergence in probability? Carefully justify all your answers.

Problem 5.
Let $\{X_n\}$ be a sequence of i.i.d. random variables in L^1 ($\mathbb{E}[|X_1|] < +\infty$). Take $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$, $n \geq 1$, and denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$.

1. Let τ be an $\{\mathcal{F}_n\}$-stopping time in L^1. Prove that S_τ is in L^1, and
$$\mathbb{E}[S_\tau] = \mathbb{E}[X_1]\mathbb{E}[\tau].$$
2. Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Denote $S_n = \sum_{k=1}^n X_k$, $S_0 = 0$, and $T = \inf\{n \geq 0 : S_n = 1\}$. Prove that $\mathbb{E}[T] = +\infty$. (One may use question 1.)

Problem 6.

Let $X_1, \dots, X_n$ be a sequence of i.i.d. random variables in L^1. Compute $\mathbb{E}[X_1 | X_1 + \dots + X_n]$.

Hint: for all $i, j \in \{1, \dots, n\}$, $\mathbb{E}[X_i | X_1 + \dots + X_n] = \mathbb{E}[X_j | X_1 + \dots + X_n]$.

Problem 7.

1. Give the definition of a standard Brownian motion.

2. Let $\{B_t\}$ and $\{\widetilde{B}_t\}$ be two independent standard Brownian motion. Let $\rho \in (0,1)$. Define, for $t \geq 0$,

$$X_t = \rho B_t + \sqrt{1-\rho^2}\,\widetilde{B}_t.$$

   Is $\{X_t\}$ a standard Brownian motion? Carefully justify.

Problem 8.

Let $\{B_t\}$ be a standard Brownian motion.

1. Fix $u > 0$. Prove that $M_t = e^{uB_t - \frac{1}{2}u^2 t}$ is a martingale (with respect to the same filtration as $\{B_t\}$).

2. Fix $a > 0$. Define $T_a = \inf\{t \geq 0 : B_t = a\}$. Prove that $T_a < +\infty$ a.s. and $\mathbb{E}[T_a] = +\infty$.

   Hint: One may find the density of T_a by considering $X_t = \max_{0 \leq s \leq t} B_s$, and noticing that $\{T_a \leq t\} = \{X_t \geq a\}$.