Complex Analysis
PhD Examination
January 2020

Answer SIX questions in detail.
State carefully any results used without proof.
Throughout, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$.

---

1. When $n > 1$ is a positive integer, evaluate the integral

$$\int_0^\infty \frac{1}{1+x^n}\mathrm{d}x.$$

2. True or false? There exists a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that $f(1/n) = f(-1/n) = 1/n^3$ for each integer $n > 1$. Prove or disprove.

3. The functions f and g are holomorphic and never zero in $\mathbb{D}$. Assume that

$$\frac{f'}{f}(2^{-n}) = \frac{g'}{g}(2^{-n})$$

for each positive integer n. Deduce as much as possible about the relationship between f and g.

4. Let f be holomorphic in $\mathbb{D} \setminus \{0\}$ and assume that its derivative f' extends holomorphically to $\mathbb{D}$. Must f itself extend holomorphically to $\mathbb{D}$? Explain.

5. Let the continuous function $f : \mathbb{D} \to \mathbb{C}$ be holomorphic except (perhaps) on the interval $[-1/2, 1/2]$. Prove that f is in fact holomorphic on $\mathbb{D}$.

6. Let the entire function f never take purely imaginary values. Without using a theorem of Picard, prove that f is constant.

7. Let the entire function f satisfy $|f(z)| \leqslant A + B|z|^M$ whenever $|z| \geqslant N$, where A, B, M, N are positive constants. Prove that f is a polynomial.

8. Let f be holomorphic in $\mathbb{D}$. Assume that for each $z \in \mathbb{D}$ there exists $n = n_z \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

9. Prove that if z and w lie in the (open) right half-plane then

$$\int_0^\infty e^{-wt}t^{z-1}\mathrm{d}t = w^{-z}\Gamma(z)$$

by integrating $e^{-\zeta}\zeta^{z-1}$ around a suitable closed contour.
