Complex Analysis
PhD Examination
August 2019

Answer SIX (out of nine) questions in detail.
State carefully any results used without proof.

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1. Let a be a point of the open unit disc $\mathbb{D}$; for $z \in \mathbb{C}$ not equal to $1/\bar{a}$ define

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Prove that ϕ_a maps $\mathbb{D}$ to itself biholomorphically and $\overline{\mathbb{D}}$ to itself homeomorphically; identify its inverse in each capacity.

2. (i) Prove that the ring $\mathcal{O}(\mathbb{D})$ of functions holomorphic in the open unit disc is an integral domain.
(ii) Let f be holomorphic in the open unit disc. If f is either even or odd, then its pointwise square f^2 is even. Is the converse true or false? Give proof or counterexample as appropriate.

3. Let f be holomorphic in the open unit disc and continuous in the closed unit disc; assume that $f(z) = 0$ for each z in the unit circle $\overline{\mathbb{D}} \setminus \mathbb{D}$ such that $\mathrm{Im} z \geqslant 0$. Prove that f is identically zero in $\overline{\mathbb{D}}$.

4. Let f be a nonconstant entire function. Prove that if $w_0 \in \mathbb{C}$ is not a value of f then for each $r > 0$ there exists $z \in \mathbb{C}$ such that $|f(z) - w_0| < r$.

5. Use the Residue Theorem to evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2} \mathrm{d}x.$$

6. Let f be holomorphic in the open unit disc, taking real values on the real axis and taking purely imaginary values on the imaginary axis. Prove that f is odd.

7. Prove that the infinite product $\prod_{n=0}^{\infty}(1 + z^{2^n})$ converges whenever $|z| < 1$. What is the value of the product? What is the nature of the convergence?

8. Define the Gamma and Beta functions, stating how they are related. When p and q have strictly positive real part, calculate the integral

$$\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, \mathrm{d}\theta$$

in terms of the Gamma function.

9. Defining $J_n(z)$ to be the coefficient of ζ^n in the Laurent expansion of the function $\exp\{\frac{1}{2}z(\zeta - \zeta^{-1})\}$ of ζ about zero, show that

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\mathrm{d}\theta.$$
