PhD Complex Analysis Exam, May 2017

Do six of eight

Throughout, $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{H} = \{\operatorname{Re} z > 0\}$. Work each problem on a separate sheet.

(1) Use Liouville's Theorem to prove the fundamental theorem of algebra.

(2) Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire and $|f(z)| = 1$ for $|z| = 1$. What can be said about f?

(3) Find
$$\int_0^\pi \frac{dt}{2 + \cos(t)}.$$

(4) Let $\mathbb{P} = \{0 < |z| < 1\}$.
 (a) Does there exist an onto analytic map $f : \mathbb{D} \to \mathbb{P}$? If so, can f be one-one?
 (b) Find a conformal map from the strip $S = \{0 < \operatorname{Im} z < 1\}$ to $\mathbb{D}$.

(5) Suppose $f : \mathbb{C} \to \mathbb{C}$ is entire, non-constant and periodic of period 1, meaning $f(z + 1) = f(z)$. Prove the entire function $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = f(z) - z$ is onto.

(6) Suppose f is analytic in an open set containing $\{|z| \le 1\}$ and $|f(z)| < 1$ for $|z| = 1$. Prove $f(z) = z$ has exactly one solution (counting with multiplicity) in $\mathbb{D}$.

(7) Let
$$\mathscr{F} = \{f : \mathbb{D} \to \mathbb{H} : f \text{ is analytic and } f(0) = 1\}.$$
Prove $\mathscr{F}$ is a normal family.

(8) Fix $0 < r < 1$ and let $\mathbb{A} = \{r < |z| < 1\}$. Show, if u is harmonic on $\mathbb{A}$, then there is an analytic function f on $\mathbb{A}$ and a real constant c such that
$$u(z) = \operatorname{Re} f(z) + c \log(|z|).$$