UF Combinatorics PhD Exam — August 2024

1. Let $h(n)$ be the number of words of length n over the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ in which an a is never immediately followed by a b. Find the ordinary generating function of for sequence $\{h(n)\}$.

2. Let $H(n)$ be the number of permutations of length n that have an even number of fixed points. Find the exponential generating function of the sequence $\{H(n)\}$.

3. Select a 132-avoiding permutation π of length n uniformly at random. Which is more likely, that n is at either end of π, or that n is somewhere between positions 2 and $n-1$ (inclusive) in π? Justify your answer.

4. Let T be a tree. Is it true that there is a vertex v in T that is part of all paths of maximum length in T? Justify your answer.

5. Prove that a graph in which all of the vertices have even degrees contains no bridges. (Recall that an edge e of a graph G is a *bridge* if $G - e$ has more connected components than G.)

6. Prove that every convex polyhedron has two faces that have the same number of vertices.

7. Find the number of all 2-element antichains in the Boolean algebra B_n.

8. Let P be a finite poset. Prove that the number of elements in a maximum chain in P equals the number of antichains in the smallest antichain cover of P. (Recall that an *antichain cover* of P is a set of antichains whose union is P.)