ANALYSIS QUALIFYING EXAM MAY 2025

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

DO SIX OF THE EIGHT PROBLEMS.

(1) For any two of the following, either give an example or explain why no example exists.

   (i) A Lebesgue measurable set $E \subseteq [0,1]$ that is closed, has positive measure, but contains no non-trivial interval.

   (ii) A Lebesgue measurable set $E \subseteq [0,1]$ of positive measure such that $E - E = \{x - y : x, y \in E\}$ contains no non-trivial interval.

   (iii) A measure space $(X, \mathscr{M})$ and a set $E \subseteq X \times X$ such that $[E]_x$, $[E]^y \in \mathscr{M}$ for all $x, y \in X$, but $E \notin \mathscr{M} \otimes \mathscr{M}$.

(2) Suppose X, Y are metric spaces. Show, if $f : X \to Y$ is continuous, then f is Borel measurable.

(3) Let $I = (-1, 1)$ and let m^* denote Lebesgue outer measure. Show, if $A \subseteq I$ and $m^*(A) + m^*(I \setminus A) = 2$, then A is Lebesgue measurable.

(4) State the Radon-Nikodym Theorem and give an example that shows the σ-finite hypothesis is needed.

(5) Suppose $\mathscr{X}$ is a Banach space and $\mathcal{M}$ and $\mathcal{N}$ are (closed) subspaces of $\mathscr{X}$. Show, if each $x \in \mathscr{X}$ has a unique representation as $x = m + n$ for some $m \in \mathcal{M}$ and $n \in \mathcal{N}$, then the mapping $\mathscr{X} \to \mathcal{M}$ sending $x = m + n$ to m is linear and bounded.

(6) Prove, if $\mathscr{X}$ is a Banach space and $\mathscr{X}^*$ is separable, then $\mathscr{X}$ is separable. Is ℓ^1 isomorphic, as a Banach space, to $(\ell^\infty)^*$?

(7) Short answer. Do two.

(i) Suppose $h : \mathbb{R} \to \mathbb{C}$ is a measurable function. If $hf \in L^1(\mathbb{R})$ for every $f \in L^4(\mathbb{R})$, what can be said about h? Explain your answer.

(ii) Does there exist a (bounded) linear functional $L : \ell^\infty(\mathbb{N}) \to \mathbb{C}$ such that $L(f) = \lim f(n)$ whenever $f \in \ell^\infty(\mathbb{N})$ and $(f(n))_n$ converges? Explain your answer.

(iii) Does there exist an inner product on $\mathbb{R}^2$ that induces the norm $\|x\| = |x_1| + |x_2|$ for $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2$. Explain your answer.

(8) Do two providing short justifications for your answers. Here $\widehat{\ }$ denotes the Fourier transform of an L^1 function and $\mathcal{F}f$ denotes the Fourier transform of an $L^2(\mathbb{R})$ function f.

(i) Define $f \in L^1(\mathbb{R})$ by $f(x) = e^{-x^4}\chi_{[1,\infty)}(x)$. Is $\widehat{f} \in L^1(\mathbb{R})$?

(ii) Recall, if $f \in L^1(\mathbb{R})$, then $\|\widehat{f}\|_\infty \leq \|f\|_1$. When does equality hold?

(iii) Does there exist $0 \neq f \in L^2(\mathbb{R})$ such that $\mathcal{F}f = e^{i\frac{\pi}{4}} f$?